3.4.1 $y''' - 2y'' + y' = 1 + xe^x \quad y(0)=0 \quad y'(0)=0 \quad y''(0)=1$

$y' = u \qquad u'' - 2u' + u = 1 + xe^x$

$r^2 - 2r + 1 = 0 \qquad r = 1, 1$

$u_1 = e^x, \qquad u_2 = xe^x$

$u = c_1 e^x + c_2 x e^x + u_p$

$u_p = 1 + (Ax+B)x^2 e^x$

$u_p' = (3Ax^2 + 2Bx + Ax^3 + Bx^2)e^x$

$u_p' = (Ax^3 + (3A+B)x^2 + 2Bx)e^x$

$u_p'' = (3Ax^2 + 2(3A+B)x + 2B + Ax^3 + (3A+B)x^2 + 2Bx)e^x$

$= (Ax^3 + (6A+B)x^2 + (6A+4B)x + 2B)e^x$

$u_p'' - 2u_p' + u_p = 1 + xe^x$

$Ax^3 + (6A+B)x^2 + (6A+4B)x + 2B$

$-2(Ax^3 + (3A+B)x^2 + 2Bx)$

$+ 0 + (Ax+B)x^2 \quad = 0 + x$

$x^3: \quad A - 2A + A = 0$

$x^2: \quad 6A+B - 2(3A+B) + B = 0$

$x: \quad 6A+4B - 4B = 1 \qquad A = \frac{1}{6}$

$x^0: \quad 2B = 0 \quad B = 0$

$u = c_1 e^x + c_2 x e^x + 1 + \frac{x^3}{6}e^x$

$u(0) = 0 \qquad u(0) = c_1 + 1 = 0 \quad c_1 = -1$

$u'(0) = 1 \qquad u'(x) = c_1 e^x + c_2 e^x + c_2 x e^x + \frac{x^2}{2}e^x + \frac{x^3}{6}e^x$

$u'(0) = 1 = -1 + c_2 + 0 \quad c_2 = 2$

$y' = u(x) = -e^x + 2xe^x + 1 + \frac{x^3}{6}e^x$

$y = \int_0^x -e^z + 2ze^z + 1 + \frac{z^3}{6}e^z \, dz + 0$

3.4.2 $e^x y''' + \sin x\, y'' + \cos x \cdot y' + x^6 \cdot y = 0 \quad y(0)=0,\ y'(0)=0,\ y''(0)=0$

$\Rightarrow$ homogeneous eq w/ homogeneous bcs $\Rightarrow$ $y(x)=0$

3.4.3 $xy'' - (x+1)y' + y = x^2 e^{2x} \quad y(1)=0 \quad y'(1)=e$

$y_1 = x+1 \qquad y_2 = v y_1, \quad v' + \left(\frac{2y_1'}{y_1} + \frac{a_1}{a_0}\right) v = 0.$

$$v' + \left(2 \cdot \frac{1}{x+1} + \frac{-(x+1)}{x}\right) v = 0.$$

$$\frac{v'}{v} = -\frac{2}{x+1} + 1 + \frac{1}{x}$$

$$\ln v = -2\ln x+1 + x + \ln x = \ln \frac{x}{(x+1)^2} + x$$

$$v = e^x \frac{x}{(x+1)^2} \qquad y_2 = \frac{x}{x+1} e^x = \left(1 - \frac{1}{1+x}\right)e^x$$

$$y_p = \left(-\int \frac{\frac{x}{x+1}e^x \cdot xe^{2x}}{W}\, dx \right) y_1 + \left(\int \frac{(x+1)\cdot xe^{2x}}{W} \right) y_2$$

$$W = \begin{vmatrix} x+1 & \frac{x}{x+1}e^x \\ 1 & \frac{e^x}{(1+x)^2} + \left(1 - \frac{1}{1+x}\right)e^x \end{vmatrix}$$

$$= \frac{e^x}{1+x} + (1+x-1)e^x - \frac{x}{x+1}e^x$$

$$= e^x\left(\frac{1-x}{1+x} + x\right) = e^x \frac{1-x+x+x^2}{1+x} = e^x \frac{1+x^2}{1+x}$$

$$y = c_1(x+1) + c_2 \frac{x}{1+x} e^x + y_p$$

3.4.4 $y'' + 2xy' = e^{-x^2} \quad y(0)=1, \; y'(0)=2$

$u = y' \quad u' + 2xu = e^{-x^2}$

$\mu' = 2x\mu \qquad \frac{\mu'}{\mu} = 2x \quad \ln\mu = x^2 \quad \mu = e^{x^2}$

$(e^{x^2}u)' = 1 \qquad e^{x^2}u = x + c$

$u = xe^{-x^2} + c\cdot e^{-x^2} = xe^{-x^2} + 2e^{-x^2}$

$u(0) = 2 = c$

$y' = xe^{-x^2} + 2e^{-x^2}$

$y = \int_0^x ze^{-z^2} + 2e^{-z^2}\,dz + 1$

$y(x) = -\frac{e^{-x^2}}{2} + 2\int_0^x e^{-z^2}\,dz + 1$

3.4.5 $(x-1)y'' - xy' + y = (x-1)^2 \qquad 0<x<\frac{1}{2}, \quad y(0) = y(\tfrac{1}{2}) = 0$

$y_1 = x, \; y_2 = x\cdot u, \qquad u' = v$

$v' + \left(\frac{2}{x} + \frac{-x}{x-1}\right)v = 0$

$\frac{v'}{v} = -\frac{2}{x} + \frac{x}{x-1} = -\frac{2}{x} + 1 + \frac{1}{x-1}$

$\ln v = \ln x^{-2} + x + \ln(x-1) = \ln\frac{x-1}{x^2} + x$

$v = e^x\cdot\frac{x-1}{x^2}, \quad y_2 = \frac{x-1}{x}e^x$

$u' = e^x\frac{x-1}{x^2} \quad u = \int \frac{x-1}{x^2}e^x\,dx, \quad u = -\frac{e^x}{x}$

$y_2 = -\frac{e^x}{x}\cdot x = -e^x$

$y_H = c_1x + c_2e^x \qquad y_H(0) = 0 = y_H(\tfrac{1}{2}) \qquad c_1 = 0, \; c_2 = 0$

$y_p = \left(-\int \frac{e^x\cdot(x-1)}{W}dx\right)x + \left(\int \frac{x(x-1)}{W}dx\right)e^x$

3.4.6 $x^2y'' - 2xy' + 2y = 1$ $\quad 1<x<2, \quad y(1)=1 \quad y(2)=0$

$r(r-1) - 2r + 2 = 0$ $\quad y_1 = x \quad y_2 = x^2$

$r^2 - 3r + 2 = 0$

$(r-1)(r-2)=0, \quad r=1,2$ $\quad W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

$$y_p = \left(-\int \frac{x^2 \cdot \frac{1}{x^2}}{x^2} dx \right) x + \left(\int \frac{x \frac{1}{x^2}}{x^2} dx \right) x^2$$

$$= \frac{1}{x} \cdot x + \left(-\frac{1}{2} \frac{1}{x^2} \right) x^2$$

$$= \frac{1}{2}$$

no non trivial homogeneous solution.

$y = c_1 x + c_2 x^2 + \frac{1}{2}$

$y(1) = 1 = c_1 + c_2 + \frac{1}{2}$ $\quad -2 = 2c_2 - \frac{1}{2}$

$y(2) = 0 = 2c_1 + 4c_2 + \frac{1}{2}$ $\quad -\frac{3}{2} = 2c_2, \quad c_2 = -\frac{3}{4}$

$c_1 = -c_2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

$y = \frac{5}{4}x - \frac{3}{4}x^2 + \frac{1}{2}$

3.4.7. $y'' = 2xe^{x^2}$ $\quad y(0)=0 \quad y'(0)=2$

$u = y'$

$u' = 2xe^{x^2}$, $\quad u = e^{x^2} + c$ $\quad u(0) = e^0 + c = 2 \Rightarrow c = 1$

$y' = e^{x^2} + 1$

$$y = \int_0^x e^{z^2} dz + x + c \qquad c = 0$$

$$y(x) = \int_0^x e^{z^2} dz + x$$

3.4.8 $4y'' + y = x \qquad -\pi < x < \pi \qquad y(-\pi) = 0 = y(\pi)$

$4r^2 + 1 = 0 \qquad r = \pm \frac{1}{2} i$

$y_H = C_1 \cos\frac{x}{2} + C_2 \sin\frac{x}{2} \Rightarrow C_2 = 0$ one nontrivial y_H

$$\Rightarrow \int_{-\pi}^{\pi} \cos\frac{x}{2} \cdot x \, dx = x \, 2 \cdot \sin\frac{x}{2} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2\sin\frac{x}{2} \, dx$$

$$= 4\pi + 4\cos\frac{x}{2} \Big|_{-\pi}^{\pi} = 4\pi \neq 0$$

$\Rightarrow$ no solution

3.4.9. $y'' = \sin^2 x \qquad 0 < x < 1, \quad y(0) = 0, \quad y'(1) = 1$

$y_H'' = 0 \qquad y_H = ax + b \qquad a = 0, \ b = 0$

$1 - 2\sin^2 x = \cos 2x \qquad \frac{1 - \cos 2x}{2} = \sin^2 x$

$y_p'' = \frac{1}{2} - \frac{1}{2}\cos 2x$

$y_p = \frac{1}{4}x^2 + \alpha x + \beta + A\cos 2x + B\sin 2x$

$y_p'' = \frac{1}{2} - 4A\cos 2x - 4B\sin 2x$

$-4A = -\frac{1}{2}, \ -4B = 0 \Rightarrow A = \frac{1}{8}, \ B = 0$

$y = \frac{1}{4}x^2 + \alpha x + \beta + \frac{1}{8}\cos 2x$

$y' = \frac{1}{2}x + \alpha - \frac{1}{4}\sin 2x, \quad y'(1) = \frac{1}{2} + \alpha - \frac{\sin 2}{4} = 1$

$\alpha = \frac{\sin 2}{4} + \frac{1}{2}$

$y(0) = 0 = \beta + \frac{1}{8} = 0 \quad \beta = -\frac{1}{8}$

$y = \frac{1}{4}x^2 + \left(\frac{1}{2} + \frac{\sin 2}{4}\right)x - \frac{1}{8} + \frac{1}{8}\cos 2x$

3.4.10 $y'' = \cos x \qquad 0 < x < \pi, \qquad y(0) = 0 \quad y'(0) = 0$

$y = ax + b - \cos x$

$y(0) = b - 1 = 0 \quad b = 1$

$y'(0) = a - \sin 0 = 0 \quad a = 0$

$\boxed{y = 1 - \cos x}$

3.4.11 $y'' - \pi^2 y = \sin \pi x \qquad 0 < x < 1, \quad y(0) = 0 \quad y(1) = 0.$

$y_H'' - \pi^2 y_H = 0 \qquad y_H = a e^{\pi x} + b e^{-\pi x} \qquad a + b = 0$

$a e^{\pi} + b e^{-\pi} = 0 \implies a = b = 0$

$y = A e^{\pi x} + B e^{-\pi x} + y_p.$

$y_p = \alpha \sin \pi x + \beta \cos \pi x \quad y_p'' = -\pi^2 (\alpha \sin \pi x + \beta \cos \pi x)$

$-2\pi^2 \cdot (\alpha \sin \pi x + \beta \cos \pi x) = \sin \pi x$

$\alpha = -\frac{1}{2\pi^2} \quad \beta = 0$

~~$y = A e^{\pi x} + B e^{-\pi x} - \frac{1}{2\pi^2} \sin \pi x$~~

$y(0) = 0 = A + B$

~~$y'(0) = \pi(A - B) - \frac{1}{2\pi} = 0$~~

~~$A - B = \frac{1}{2\pi^2} \qquad A = -B$~~

~~$2A = \frac{1}{2\pi^2}, \quad A = \frac{1}{4\pi^2} = -B$~~

~~$y = \frac{1}{4\pi^2}(e^{\pi x} - e^{-\pi x}) - \frac{1}{2\pi^2} \sin \pi x$~~

3.4.12 $y'' + \pi^2 y = 1 - x^2 \qquad 0 < x < 1, \quad y(0) = 0 \quad y(1) = 0.$

$y_H = \sin \pi x \qquad \int_0^1 (1 - x^2) \sin \pi x \, dx = -\frac{1}{\pi} \cos \pi x \Big|_0^1 - \int_0^1 x^2 \sin \pi x \, dx$

$$\int_0^1 x^2 \sin \pi x \, dx = x^2\left(-\frac{1}{\pi}\cos \pi x\right)\Big|_0^1 + \int_0^1 \frac{2x}{\pi}\cos \pi x \, dx$$

$$= +\frac{1}{\pi} + \frac{2}{\pi}\int_0^1 x\cos \pi x \, dx$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\left(\frac{x}{\pi}\sin \pi x\Big|_0^1 - \frac{1}{\pi}\int_0^1 \sin \pi x \, dx\right)$$

$$= \frac{1}{\pi} + \frac{2}{\pi}\left(+\frac{1}{\pi}\cdot\frac{1}{\pi}\cos \pi x\Big|_0^1\right)$$

$$= \frac{1}{\pi} + \frac{2}{\pi^3}(-2) = \frac{1}{\pi} - \frac{4}{\pi^3}$$

$\therefore \int_0^1 f(x)\, y_H \, dx \neq 0 \Rightarrow$ no solution

3.4.13. $y'' = \sin x \quad 0 < x < 1, \quad y(0) = 0 \quad y(1) = 1$

$y_H = ax + b \quad a = 1,\ b = 0 \quad y_H = x$

$$\int_0^1 x \sin x \, dx = -x\cos x\Big|_0^1 + \int_0^1 \cos x \, dx$$

$$= -\cos 1 + \sin 1 \neq 0 \Rightarrow \text{no solution}$$

3.4.14. $x^2y'' - 4xy' + 6y = 2x \quad 1 < x < 2 \quad y(1) = 1 \quad y(2) = 2$

$r(r-1) - 4r + 6 = 0$

$r^2 - 5r + 6 = 0, \ r = 2, 3$

$y_H = c_1 x^2 + c_2 x^3 \quad y_H(1) = 1 = c_1 + c_2 \qquad 4c_2 = -2, \ c_2 = -\frac{1}{2}$

$y_H(2) = 2 = 4c_1 + 8c_2 \qquad c_1 = \frac{3}{2}$

$$\int_1^2 \left(\frac{3}{2}x^2 - \frac{1}{2}x^3\right) 2x \, dx = x^3 - \frac{1}{4}x^4\Big|_1^2$$

$$= 8 - 1 - \frac{1}{4}(16 - 1)$$

$$= 7 - \frac{15}{4} \neq 0 \quad \text{no solution}$$

3.4.15 $9y'' + y = xe^{-x^2}$ $\quad -3\pi < x < 3\pi \quad y(3\pi) = 0 = y(3\pi)$

$y_H = C_1 \cos\frac{x}{3} + C_2 \sin\frac{x}{3} \qquad C_1 = 0 \quad y_H = \sin\frac{x}{3}$

$$\int_{-3\pi}^{3\pi} \sin\frac{x}{3} \cdot xe^{-x^2} dx = \cancel{\sin\frac{x}{3} \frac{1}{2} e^{-x^2}}\Big|_{-3\pi}^{3\pi} - \int_{-3\pi}^{3\pi} \frac{1}{6} \cos\frac{x}{3} e^{-x^2} dx$$

$$= -\frac{1}{6}\int_{-3\pi}^{3\pi} \cos\frac{x}{3} \cdot e^{-x^2} dx \neq 0$$

no solution

3.4.16 $y'' + y = f(x) \quad 0 < x < 2\pi \quad y(0) = 0 \quad y(2\pi) = 0$

$y_H = \sin x \Rightarrow$ no solution if $\int_0^{2\pi} \sin x \, f(x)\, dx \neq 0$

3.4.17. $-y'' = f(x) \quad 0 < x < 1, \quad y(0) = \alpha, \; y(1) = \beta$

$y_H = ax + b = \beta x + \alpha$

$\int_0^1 f(x)(\beta x + \alpha)\, dx = 0$ infty. many solns.

$\neq 0$ no solution