Parameterized complexity and improved inapproximability for computing the largest j-simplex in a V-polytope

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Abstract

We consider the problem of computing the squared volume of the largest j-simplex contained in an n-dimensional polytope presented by its vertices (a V-polytope). We show that the related decision problem is $W[1]$-complete, with respect to the parameter j. We also improve the constant inapproximability factor given in [Packer, 2004, Discrete Applied Mathematics, 134], by showing that there are constants $\mu < 1, c > 1$, such that it is NP-hard to approximate within a factor of $c^{\mu n}$ the volume of the largest $\lfloor \mu n \rfloor$-simplex contained in an n-dimensional polytope with $O(n)$ vertices.

1 Introduction

A polytope is the convex hull of a finite set of points in the Euclidean space. A polytope can be presented either as a finite set of linear inequalities (H-representation), or as the convex hull of its vertices (V-representation). It is well known that for a given polytope, the size of its V-representation can be exponentially bigger than the size of its H-representation, and vice-versa. Because of that, the two encodings are not equivalent when studying questions of efficient computability. A more general but weaker way of presenting a polytope is by means of a membership oracle which when presented with a point, it answers whether the point is in the polytope or not. A j-simplex is the convex hull of $j+1$ affinely independent points in the Euclidean space. For a full exposition of basic notions of convex geometry we refer the reader to [13].

In this paper we focus on the problem of computing the volume of the largest j-simplex in a polytope given in its V-representation. Besides its theoretical importance in convex geometry, the problem has applications in situations where one is given a set of data points in a high dimensional space, and there is reason to believe that the ideal shape of the cloud of the data points is that of a simplex [11].

1.1 Previous results

The largest 1-simplex of a polytope is its diameter. The diameter of a V-polytope can be computed trivially in polynomial time, by computing the distances of all pairs of vertices. Brieden showed that it is NP-hard to approximate within any constant the diameter and several other functionals of H-polytopes [2]. Brieden et al. gave a deterministic polynomial time algorithm which approximates the diameter of a convex body given by a membership oracle, within a factor of $O(\sqrt{n/\log n})$. They also showed that unless P=NP, this is up to a constant the best approximation that can be achieved, even if randomization is allowed [3]. This is the best algorithm known for H-polytopes. The problem of finding the j-simplex of largest volume in an n-dimensional H-polytope, for $j \leq \mu n$, where $\mu < 1$ is any constant, was shown NP-hard by Gritzmann et al. [7]. The NP-hardness was extended to the special case $j = n$ by Packer [11]. Brieden et al. gave a polynomial time algorithm that approximates within $O((cn)^j)$ the largest j-simplex contained in a convex body given by an oracle [4]. Again, this is the best algorithm known for H-polytopes.

The problem of computing the squared volume of the largest j-polytope in an n-dimensional V-polytope was shown NP-hard in [7], for $j = \Omega(n^{1/k})$, where k is some natural number. The result was improved by Packer who showed an inapproximability factor of 1.09 [12]. Packer also gave a polynomial time algorithm that finds a simplex whose squared volume is within a factor of $O((cj)^j)$ of the maximum, for some constant c. The simplest class of V-polytopes for which these hardness results hold are polytopes whose coordinates are restricted to $\{-1, 1, 0\}$, and as noted in [7], it is an open question whether some of the hardness results can be extended to the even simpler class of $\{0, 1\}$ V-polytopes. Also, as discussed in [7] and [12], there are no hardness results for $j = o(n^{1/k})$.

1.2 Our contributions

We characterize the complexity of the problem for arbitrary functions $j = g(n)$. We show that with j considered as a parameter, the related decision problem is W[1]-complete. It is believed that problems in this class are not fixed parameter tractable, in other words they don't have algorithms with complexity of the form $f(j)poly(n)$, where f is any computable function. We refer the reader to [5] for standard notions of parameterized complexity. We also show that there are constants $\mu < 1, c > 1$, such that it is NP-hard to approximate the volume of the largest $\lfloor \mu n \rfloor$-simplex, within a factor of $c^{\mu n}$. Our results remain true for $\{0, 1\}$ polytopes. We will state our results in terms of the squared volume, because it is always a rational number when the coordinates of the polytope are rational numbers.

2 Background results

We first review some facts concerning volumes of simplices. Proofs or references to the proofs can be found in [7],[11].

Theorem 2.1 *Among all the largest j-simplices contained in a polytope P, there is a largest simplex S whose vertices are also vertices of P.*

Theorem 2.2 *Suppose that S is a j-simplex in $\mathbb{R}^n$ with vertices $v_1, \ldots, v_{j+1}$. Let $\hat{M} = (\hat{M}_{i,k})$ denote the $(j+1) \times (j+1)$ matrix given by $\hat{M}_{i,k} = ||v_i - v_k||_2^2$. Then*

$$2^j (j!)^2 vol^2(S) = |det(M)|$$

where M is the $(j+2) \times (j+2)$ matrix obtained by bordering $\hat{M}$ with a top row $(0, 1, \ldots, 1)$ and a left column $(0, 1, \ldots, 1)^T$.

The matrix M in the last Theorem is known as the Menger matrix of the simplex.

We will derive our results via a straightforward reduction from the set packing problem. An instance of the set packing problem consists of a collection C of m sets over a domain of n elements, and asks for the maximum number of mutually disjoints sets in C.

Theorem 2.3 *Deciding whether a family of sets contains j mutually disjoint sets is $W[1]$-complete, with respect to the parameter j. The problem remains $W[1]$-complete even when the sets in C contain the same number of elements.*

Proof. The usual representation of this problem consists of an $n \times m$ matrix whose rows correspond to the elements, the columns to the sets, and the entries indicate membership of the elements in the sets. So, the size of the instance is $s = O(mn)$. It is well known that the problem in this representation is $W[1]$-complete [5]. In other words, if there is a function f such that the problem can be decided in time $f(j)poly(s)$, then $W[1]$ is fixed parameter tractable.

From any instance C of the problem, we can easily construct an instance C' such that each set in C' contains the same number of elements, and C contains j mutually disjoint sets if and only if C' contains j mutually disjoint sets. This can be done by forming $S' \in C'$ from each $S \in C$, by appropriately padding S with extra elements that are contained only in S'. The new instance has size $s' = O(n^2 m)$. So, if C' can be decided in time $f(j)poly(s')$, then C can also be decided in time $f(j)poly(s)$ for some function f. This completes the proof. □

We will also use, properly adapted, an inapproximability result of Hazan et al. [8]. Let C be a collection of sets over a domain of n elements, with each set containing exactly 3 elements, and $|C| \geq kn$. Furthermore assume that either (i) C contains a set packing of size $\lfloor \mu n \rfloor$, or (ii) C does not contain a set packing of size greater than $\lceil \mu n \ln 3/3 \rceil$. Let $I_{\mu,\kappa}$ be the set of instances of set packing satisfying the above properties. Then Hazan et al. show the following.

Theorem 2.4 *There are constants $\mu < 1, \kappa > 1$ such that it is NP-hard to decide whether an instance from $I_{\mu,\kappa}$ satisfies (i) or (ii).*

We note that when (ii) is true, for every $C' \subseteq C$ with $|C'| = \mu n$, we can number the sets in C', so that S_{2i-1} and S_{2i} intersect, for $1 \leq i \leq \lfloor n(\mu - (\mu \ln 3/3))/2 \rfloor$.

3 Main results

3.1 Reduction from Set Packing

We are given a collection C over a domain $x_1, \ldots, x_n$ of n elements. We will assume that each set in C contains exactly t elements. We construct a polytope P in $\mathbb{R}^n$ as follows. For each set $S \in C$ we define a point v_S by $v_S(i) = 0$ if $x_i \notin S$, and $v_S(i) = 1$ if $x_i \in S$. We let P be the convex hull of these points. Note that the coordinates of the vertices of P are restricted to $\{0, 1\}$.

3.2 Parameterized Complexity

Theorem 3.1 *Given an n-dimensional V-polytope P, and $\lambda \in \mathbb{Q}$, the problem of deciding whether there is a j-simplex Q contained in P, such that $vol^2(Q) \geq \lambda$, is $W[1]$-complete.*

Proof. We construct the polytope P as described in subsection 3.1. By Theorem 2.1, to find the volume of the largest j-simplex in P, it is enough to consider the j-simplices defined by all the possible sets of $j+1$ vertices of P. When two sets S_i, S_j are disjoint, we have $||v_{S_i} - v_{S_j}||_2^2 = 2t$. When S_i and S_j intersect, we have $||v_{S_i} - v_{S_j}||_2^2 \leq 2t - 2$. If C contains a collection of $j+1$ mutually disjoint sets, the polytope P contains the regular simplex Q_R of edge length $\sqrt{2t}$. This simplex has the maximum volume among all simplices with edge lengths upper bounded by $\sqrt{2t}$ ([13],[7]). The squared volume of Q_R is $(2t)^j(j+1)/2^j(j!)^2$. Therefore, by Theorem 2.3, it is $W[1]$-hard to decide whether there is $Q \in P$ such that $vol^2(Q) \geq vol^2(Q_R)$. The completeness follows from the fact that given any $j+1$ vertices it can be easily verified whether they form the regular j-simplex. □

3.3 Inapproximability

In order to make the idea behind our inapproximability result more clear, we give the main part of the proof in this subsection. In the proof we will need Lemma 3.5 which is a purely algebraic statement proven separately in the following subsection.

Theorem 3.2 *There are constants $\mu < 1$ and $c, k > 1$, such that if P is an n-dimensional V-polytope with kn vertices, it is NP-hard to approximate within a factor of c^j the squared volume of the largest j-simplex contained in P , with $j = \lfloor \mu n \rfloor$.*

Proof. Given an instance of the set packing with the properties described in Theorem 2.4, we construct a polytope P as described in subsection 3.1. We fix μ, k to be the constants appearing in the statement of Theorem 2.4, and let $j = \mu n$. For notational simplicity let us assume that j is an integer. Note that P has kn vertices. By Theorem 2.1, to find the volume of the largest j-simplex in P, it is enough to consider the j-simplices defined by all the possible sets of $j+1$ vertices of P. Let $Q_{\max}$ be the largest j-simplex contained in P. If $Q_{\max}$ is not the regular j-simplex Q_R of squared edge length 6, then, by Theorem 2.4, we can number the sets/vertices so that the Menger matrix $M_{Q_{\max}}$ satisfies $M_{Q_{\max}}(2i, 2i+1) \leq 4$ for $i = 1, \ldots, \lfloor j/t \rfloor$, where $t = 2/(\mu - (\mu \ln 3/3))$.

To prove the theorem we need to show that there exists a constant $c > 1$ such that

$$\frac{vol^2(Q_R)}{vol^2(Q_{\max})} \geq c^j.$$

Then, since $vol^2(Q_R)$ is explicitly known, if we were able to compute the volume within a c^j factor, we would be able to decide whether the polytope contains Q_R, which corresponds to a set packing of size j.

If we scale down all the edges of $Q_R, Q_{\max}$ by a factor of $\sqrt{6}$, the ratio of their volumes does not change. So we may assume that the maximum edge length Q_R and $Q_{\max}$ is 1, and $M_{Q_{\max}}(2i, 2i+1) \leq 2/3$, for $i = 1, \ldots, j/t$. In this case $det(M(Q_R)) = j + 1$, and

$$\frac{vol^2(Q_R)}{vol^2(Q_{\max})} = |\frac{det(M_{Q_R})}{det(M_{Q_{\max}})}| = |\frac{j+1}{det(M_{Q_{\max}})}|.$$

So, it is sufficient to show that there is a constant $d < 1$, such that $|det(M_{Q_{\max}})| \leq d^j$. By substituting $n = j$, $c = 2/3$ and $k = 2/(\mu - (\mu \ln 3/3))$ in Lemma 3.5, the proof is completed. □

3.4 Technical Lemmas

Lemma 3.3. *Let $\lambda_1, \ldots, \lambda_n$ be positive numbers, with $\sum_{i=1}^{n} \lambda_i \leq n$. If $\prod_{i=1}^{\lfloor \epsilon n \rfloor} \lambda_i = c^n$ where $\epsilon, c < 1$ are constants, then $\prod_{i=1}^{n} \lambda_i < d^n$, where $d < 1$ is a constant depending only on c, ϵ.*

Proof. For notational simplicity, let us assume that ϵn is an integer. We seek to maximize

$$\prod_{i=\epsilon n+1}^{n} \lambda_i \text{ subject to } \sum_{i=\epsilon n+1}^{n} \lambda_i \leq n - \sum_{i=1}^{\epsilon n} \lambda_i.$$

The maximum is achieved when $\sum_{i=1}^{\epsilon n} \lambda_i$ is minimum subject to the condition on the corresponding product. By standard facts, this minimum is achieved when the numbers $\lambda_1, \ldots, \lambda_\epsilon$ are equal. Hence,

$$\sum_{i=1}^{\epsilon n} \lambda_i \geq c^{1/\epsilon} \epsilon n.$$

Therefore, we seek to maximize

$$\prod_{i=\epsilon n+1}^{n} \lambda_i \text{ subject to } \sum_{i=\epsilon n+1}^{n} \lambda_i \leq n(1 - \epsilon c^{1/\epsilon}).$$

By standard facts, the maximum is achieved when the numbers $\lambda_{\epsilon n+1}, \ldots, \lambda_n$ are equal. Hence,

$$\prod_{i=\epsilon n+1}^{n} \lambda_i \leq \left(\frac{1 - \epsilon c^{1/\epsilon}}{1 - \epsilon}\right)^{(1-\epsilon)n} \Rightarrow \prod_{i=1}^{n} \lambda_i \leq \left(c\left(\frac{1 - \epsilon c^{1/\epsilon}}{1 - \epsilon}\right)^{(1-\epsilon)}\right)^n.$$

To upper bound the value of the base of the exponential above, we define the function

$$f(c,\epsilon) = c\left(\frac{1-\epsilon c^{1/\epsilon}}{1-\epsilon}\right)^{(1-\epsilon)}.$$

We have

$$\frac{\partial f}{\partial c} = \frac{(1-c^{1/\epsilon})\left(\frac{1-\epsilon c^{1/\epsilon}}{1-\epsilon}\right)^{-\epsilon}}{1-\epsilon}.$$

It can be seen that the value of the derivative is positive for any $\epsilon, c < 1$. Thus the function is increasing in c, which in combination with the fact that $f(1,\epsilon) = 1$, proves that $f(c,\epsilon) < d$ for some constant $d < 1$, when $c, \epsilon < 1$. □

Lemma 3.4. *Let A be a real $n \times n$ positive definite matrix, with eigenvalues $0 \leq \lambda_1 \leq \ldots \leq \lambda_n$. For $m \leq n$, let X be an $n \times m$ matrix with minimum singular value σ. Let x_i denote the i^{th} column of X. Then*

$$\prod_{i=1}^{m} x_i^T A x_i \geq \sigma^{2m} \prod_{i=1}^{m} \lambda_i.$$

Proof. The singular value decomposition for X gives $X = U^T \Sigma V$, where U is a $n \times m$ matrix with orthonormal columns, V is a $m \times k$ unitary matrix, and Σ is a $m \times m$ diagonal matrix, with positive entries larger than σ. Let $B = X^T A X$. B is a positive definite matrix. We have $B = V^T \Sigma U A U^T \Sigma V$. Let $0 \leq \mu_1 \leq \ldots \leq \mu_m$ be the eigenvalues of UAU^T. By Cauchy's interlacing theorem (see [14], Corollary 4.4) we get $\mu_i \geq \lambda_i$. We have

$$\prod_{i=1}^{m} x_i^T A x_i = \prod_{i=1}^{m} B_{ii} \geq det(B) = det^2(\Sigma) det^2(V) det(UAU^T) \geq \sigma^{2m} \prod_{i=1}^{m} \mu_i \geq \sigma^{2m} \prod_{i=1}^{m} \lambda_i.$$

The first inequality is Hadamard's inequality for positive definite matrices (see [9], sections 7.7-8). □

Lemma 3.5. *Let $k > 1, c < 1$ be any fixed constants. Let M be the $n \times n$ Menger matrix of a simplex, satisfying (i) $M(i,i) = 0$, (ii) $M(i,j) \leq 1$, and (iii) $M(2i, 2i+1) \leq c$, for $1 \leq i \leq \lfloor n/k \rfloor$. Then, there is a constant $d < 1$, depending only on k, c such that $|det(M)| \leq d^n$.*

Proof. In the proof we will use basic properties of eigenvalues of symmetric matrices. We refer the reader to [9], [10]. For notational simplicity we assume n/k is an integer. The Menger matrix of any simplex has only one positive eigenvalue [6], [1]. Let $\lambda_1 \leq \ldots < \lambda_n$ be the eigenvalues of M. Each row sum in M does not exceed $n-1$, and thus we have $\lambda_n \leq n-1$ by Gerschgorin's theorem. We have

$$tr(M) = \sum_{i=1}^{n} \lambda_i = 0 \Rightarrow \sum_{i=1}^{n-1} |\lambda_i| \leq \lambda_n \leq n-1. \quad (1)$$

Let $\gamma^2 = (\sum_{i,j} M_{i,j})/n(n-1)$, a parameter depending only the given matrix M. Note that $\gamma < 1$. Let us assume that the following claim is true.

Claim A. There are constants $\gamma_1 < 1$ and $t < 1$, depending only on c, k, such that if $\gamma > \gamma_1$, M has n/k eigenvalues with product smaller than t^n.

If $\gamma > \gamma_1$, and since k is a constant, Claim A, inequality 1, and Lemma 3.3 imply that there is a constant $t_1 < 1$ depending only on c, k, such that the absolute value of the product of the negative eigenvalues is smaller than t_1^n.

If $\gamma \leq \gamma_1$ we have

$$\sum_{i=1}^{n} \lambda_i^2 = \sum_{i,j} M_{i,j}^2 \leq \sum_{i,j} M_{i,j} \leq \gamma^2 n(n-1) \leq \gamma_1^2 n(n-1) \Rightarrow \lambda_n \leq \gamma_1 n.$$

Since $\sum_{i=1}^{n} \lambda_i = 0$, we get $\sum_{i=1}^{n-1} |\lambda_i| \leq \gamma_1 n$. Subject to this condition, we have

$$\prod_{i=1}^{n-1} |\lambda_i| \leq \left(\frac{\gamma_1 n}{n-1}\right)^{n-1}.$$

Recall that the determinant of a matrix equals the product of its eigenvalues, and that $\lambda_n < n$. Hence, assuming the correctness of Claim A, for any value of γ, if we let $1 > d > \max\{t_1, \gamma_1/2\}$, we have $det(M) < d^n$ and the Theorem follows.

We now need to prove Claim A. Let p denote the unit norm eigenvector corresponding to λ_n. If e is the unit vector with equal entries, then $e^T M e = \gamma^2(n-1)$. Since $p^T M p \leq n-1$, we get $|e^T p| \geq \gamma$. For $1 \leq i \leq n/k$, let v_i be the unit vector with zero entries, with the exception of $v_i(2i) = 1/\sqrt{2}$ and $v_i(2i+1) = -1/\sqrt{2}$. Note that $v_i^T v_j = e^T v_i = 0$. Also, we have $0 > v_i^T M v_i > -c_1$, where $c_1 < 1$ is a constant which depends only on c.

In general, for each i, we can write

$$v_i = x_i + \alpha_i p, \tag{2}$$

where $x_i^T p = 0$. We have $p^T v_i = \alpha_i$. Then, we have $p = \sum_{i=1}^{n/k} \alpha_i v_i + (p^T e)e + v$, where $v^T v_i = v^T e = 0$. Taking the norm of the two parts of the equality, we get that

$$\sum_{i=1}^{n/k} \alpha_i^2 \leq 1 - \gamma^2. \tag{3}$$

Let $\gamma_0 < 1$ satisfy $(1 - \gamma_0^2)k = (1 - c_1)/2$. Then, if $\gamma > \gamma_0$, we have

$$k \sum_{i=1}^{n/k} \alpha_i^2 \leq (1 - c_1)/2. \tag{4}$$

It is well known that any two eigenvectors of any symmetric matrix M are orthogonal and M-orthogonal. Notice then that x_i is a linear combination of the negative eigenvectors of M. Thus, $p^T M x_i = 0$. By taking the M-inner product of both parts of equality 2, we have

$$v_i^T M v_i = x_i^T M x_i + \alpha_i^2 \lambda_n \Rightarrow |x_i^T M x_i| \leq c_1 + \alpha_i^2 n.$$

By standard facts, $\prod_{i=1}^{n/k}(c_1+\alpha_i^2 n)$ is maximized when the α_i^2's are equal to their average value $k\sum_{i=1}^{n/k}\alpha_i^2/n$. Using inequality 4, if $\gamma > \gamma_0$, we have

$$\prod_{i=1}^{n/k}|x_i^T M x_i| \leq \prod_{i=1}^{n/k}(c_1+\alpha_i^2 n) \leq ((1+c_1)/2)^{n/k}.$$

Let X be the $n \times n/k$ matrix with columns x_i, and σ be its smallest singular value. Then, since $p^T x_i = 0$, we can apply Lemma 3.4 to the positive part of $-M$, to get

$$\prod_{i=1}^{n/k}|\lambda_{n-i}| \leq \frac{\prod_{i=1}^{n/k}|x_i^T M x_i|}{\sigma^{2n/k}} \leq \left(\frac{1+c_1}{2\sigma^2}\right)^{n/k}. \tag{5}$$

Now we show a bound on σ in terms of γ. We have $\sigma^2 = \min_{||b||_2=1}||Xb||_2^2$. Let $b = (\beta_i)$ be a vector such that $\sum_i \beta_i^2 = 1$. Let $x = Xb = \sum_i \beta_i x_i$. Using equality 2, we have

$$\sum_i \beta_i v_i = x + \sum_i \beta_i \alpha_i p \Rightarrow 1 = ||x||_2^2 + (\sum_i \alpha_i \beta_i)^2.$$

To minimize $||x||_2^2$, we seek to maximize $(\sum_i \alpha_i \beta_i)^2$, subject to the conditions $\sum_i \beta_i^2 = 1$ and $\sum_i \alpha_i^2 \leq 1-\gamma^2$. By the Cauchy-Schwarz inequality, we get

$$(\sum_i \alpha_i \beta_i)^2 \leq (\sum_i \alpha_i^2)(\sum_i \beta_i^2) \leq 1-\gamma^2.$$

Thus $\sigma^2 \geq \gamma^2$.

Finally, let $1 > \gamma_1 = max\{\gamma_0, (1+(1+c_1)/2)/2\}$. Then, if $\gamma > \gamma_1$, we have $(1+c_1)/2\sigma^2 \leq (1+c_1)/(1+(1+c_1)/2) < 1$. By plugging this to inequality 5 the claim follows with $t = ((1+c_1)/(1+(1+c_1)/2))^k$.

□

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