Week 9
Chapter 10 Section 1-5

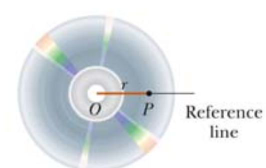
Rotation

Rigid Object

- A rigid object is one that is nondeformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible
- This simplification allows analysis of the motion of an extended object

Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin
  - A small element of the disc can be modeled as a particle at P



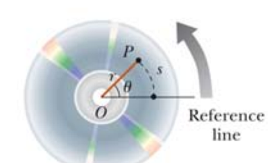
(a)

Angular Position, 2

- Point *P* will rotate about the origin in a circle of radius *r*
- ***Every*** particle on the disc undergoes circular motion about the origin, *O*
- Polar coordinates are convenient to use to represent the position of *P* (or any other point)
- *P* is located at (r, θ) where *r* is the distance from the origin to *P* and θ is the measured counterclockwise from the reference line

Angular Position, 3

- As the particle moves, the only coordinate that changes is θ
- As the particle moves through θ, it moves though an arc length s.
- The arc length and r are related:
  - $s = \theta r$



(b)

Radian

- This can also be expressed as

$$\theta = \frac{s}{r}$$

- θ is a pure number, but commonly is given the artificial unit, radian
- One radian is the angle subtended by an arc length equal to the radius of the arc
- Whenever using rotational equations, you must use angles expressed in radians

Conversions

- Comparing degrees and radians

$$1\,rad = \frac{360°}{2\pi} = 57.3°$$

- Converting from degrees to radians

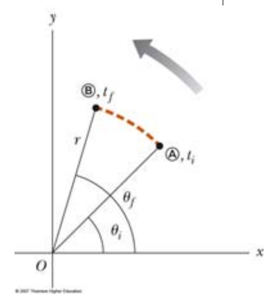
$$\theta\,(rad) = \frac{\pi}{180°}\,\theta\,(degrees)$$

Angular Displacement

- The *angular displacement* is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- This is the angle that the reference line of length *r* sweeps out



Average Angular Speed

- The *average* angular speed, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

Instantaneous Angular Speed

- The *instantaneous* angular speed is defined as the limit of the average speed as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular Speed, final

- Units of angular speed are radians/sec
  - rad/s or s^{-1} since radians have no dimensions
- Angular speed will be positive if θ is increasing (counterclockwise)
- Angular speed will be negative if θ is decreasing (clockwise)

Average Angular Acceleration

- The average angular acceleration, α, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha_{avg} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

Angular Acceleration, final

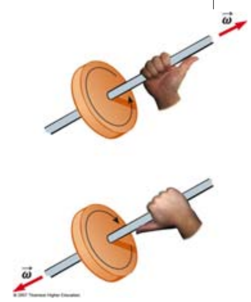
- Units of angular acceleration are rad/s² or s^{-2} since radians have no dimensions
- Angular acceleration will be positive if an object rotating counterclockwise is speeding up
- Angular acceleration will also be positive if an object rotating clockwise is slowing down

Angular Motion, General Notes

- When a rigid object rotates about a fixed axis in a given time interval, every portion on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration
  - So θ, ω, α all characterize the motion of the entire rigid object as well as the individual particles in the object

Directions, details

- Strictly speaking, the speed and acceleration (ω, α) are the magnitudes of the velocity and acceleration vectors
- The directions are actually given by the right-hand rule



Rotational Kinematics

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations
- The new model is a **rigid object under constant angular acceleration**
  - Analogous to the particle under constant acceleration model

Rotational Kinematic Equations

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\left(\theta_f - \theta_i\right)$$

$$\theta_f = \theta_i + \frac{1}{2}\left(\omega_i + \omega_f\right)t$$

all with consant α

Comparison Between Rotational and Linear Equations

TABLE 10.1

Kinematic Equations for Rotational and Translational Motion Under Constant Acceleration

Rotational Motion About a Fixed Axis	Translational Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Relationship Between Angular and Linear Quantities

- Displacements

  $s = \theta r$
- Speeds

  $v = \omega r$
- Accelerations

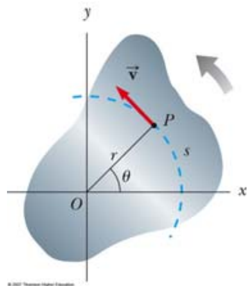
  $a = \alpha r$

- Every point on the rotating object has the same angular motion
- Every point on the rotating object does **not** have the same linear motion

Speed Comparison

- The linear velocity is always tangent to the circular path
  - Called the tangential velocity
- The magnitude is defined by the tangential speed

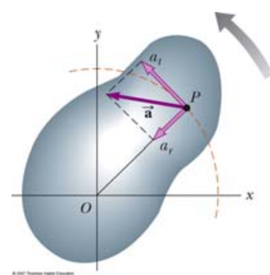
$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$$



Acceleration Comparison

- The tangential acceleration is the derivative of the tangential velocity

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha$$



Clicker Question

Alex and Brian are riding on a merry-go-round. Alex rides on a horse at the outer rim of the circular platform, twice as far from the center of the circular platform as Brian, who rides on an inner horse. When the merry-go-round is rotating at a constant angular speed, what is Alex's tangential speed?

A. twice Brian's
B. the same as Brian's
C. half of Brian's
D. four times of Brian's
E. impossible to determine

Speed and Acceleration Note

- All points on the rigid object will have the same angular speed, but not the same tangential speed
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration
- The tangential quantities depend on r, and r is not the same for all points on the object

Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
  - Therefore, each point on a rotating rigid object will experience a centripetal acceleration

$$a_C = \frac{v^2}{r} = r\omega^2$$

Resultant Acceleration

- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Rotational Motion Example

- For a compact disc player to read a CD, the angular speed must vary to keep the tangential speed constant ($v_t = \omega r$)
- At the inner sections, the angular speed is faster than at the outer sections

Rotational Kinetic Energy

- An object rotating about some axis with an angular speed, ω, has rotational kinetic energy even though it may not have any translational kinetic energy
- Each particle has a kinetic energy of
  - $K_i = \frac{1}{2} m_i v_i^2$
- Since the tangential velocity depends on the distance, r, from the axis of rotation, we can substitute $v_i = \omega_i r$

Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

- Where I is called the moment of inertia

Rotational Kinetic Energy, final

- There is an analogy between the kinetic energies associated with linear motion ($K = \frac{1}{2} mv^2$) and the kinetic energy associated with rotational motion ($K_R = \frac{1}{2} I\omega^2$)
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object
- The units of rotational kinetic energy are Joules (J)

Moment of Inertia

- The definition of moment of inertia is
  $I = \sum_i r_i^2 m_i$
- The dimensions of moment of inertia are ML^2 and its SI units are $kg \cdot m^2$
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass Δm_i

Moment of Inertia, cont

- We can rewrite the expression for I in terms of Δm
  $I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$
- With the small volume segment assumption,
  $I = \int \rho r^2 dV$
- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known

Notes on Various Densities

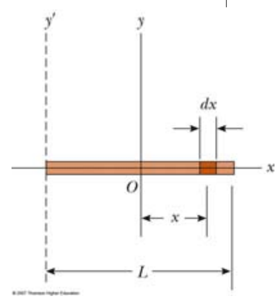
- Volumetric Mass Density → mass per unit volume: $\rho = m / V$
- Surface Mass Density → mass per unit thickness of a sheet of uniform thickness, t : $\sigma = \rho t$
- Linear Mass Density → mass per unit length of a rod of uniform cross-sectional area: $\lambda = m / L = \rho A$

Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  - $dm = \lambda dx$
- Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{12} ML^2$$

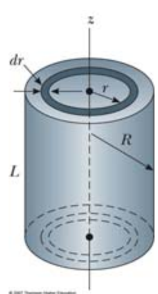


Moment of Inertia of a Uniform Solid Cylinder

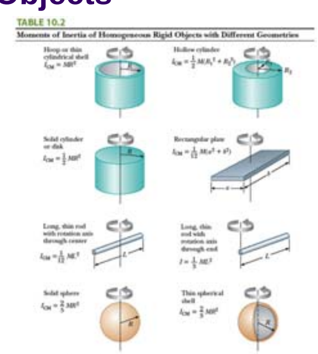
- Divide the cylinder into concentric shells with radius r, thickness dr and length L
- $dm = \rho dV = 2\pi\rho Lr\, dr$
- Then for I

$$I_z = \int r^2 dm = \int r^2 (2\pi\rho Lr\ dr)$$

$$I_z = \frac{1}{2} MR^2$$



Moments of Inertia of Various Rigid Objects



Clicker Question

A section of hollow pipe and a solid cylinder have the same radius, mass and length. They both rotate about their long central axes with the same angular speed. Which object has higher rotational kinetic energy?

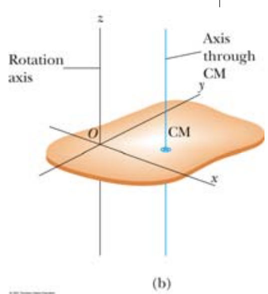
A. Solid cylinder
B. The hollow pipe
C. Same
D. Information is not sufficient to determine

Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations
- The theorem states $I = I_{CM} + MD^2$
  - I is about any axis parallel to the axis through the center of mass of the object
  - I_{CM} is about the axis through the center of mass
  - D is the distance from the center of mass axis to the arbitrary axis

Parallel-Axis Theorem Example

- The axis of rotation goes through O
- The axis through the center of mass is shown
- The moment of inertia about the axis through O would be $I_O = I_{CM} + MD^2$



(b)

Moment of Inertia for a Rod Rotating Around One End

- The moment of inertia of the rod about its center is

$$I_{CM} = \frac{1}{12}ML^2$$

- D is ½ L
- Therefore,

$$I = I_{CM} + MD^2$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$