Lecture 2 Basic MHD Equations

MHD = Magnetohydrodynamics

Maxwell's Equation:

$$\begin{cases} \vec{\nabla} \times \vec{B} = \mu\vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{E} = \frac{\rho^*}{\varepsilon} \end{cases}$$

$$\vec{H} = \frac{\vec{B}}{\mu}$$

$\vec{D} = \varepsilon\vec{E}$ Electric displaceme nt

ρ : charge density

$\vec{j}$: current density

$\varepsilon_0 = 8.854 \times 10^{-12} Fm^{-1}$ magnetic permittivity

$\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ magnetic permeability

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 m/s$ speed of light

If v << c non-relativistic

$$\vec{\nabla} \times \vec{B} = \mu\vec{j}$$

$\lambda_d = (\frac{kT}{4\pi n e^2})^{\frac{1}{2}} << l_{scale}$ in plasma

OHM' s law : $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$

$\vec{E}$ in the frame of reference moving with plasma

σ electric conductivity $ohm\ m^{-1}$

A generalization of Ohm’s law

3-fluid model for $\begin{matrix} e^- \\ n_e \end{matrix}$, $\begin{matrix} p^+ \\ n_p \end{matrix}$ and neutral atom n

$$n_e e(\vec{E}_0 + \frac{\nabla \vec{p}_e}{n_e e}) = \frac{m_e}{c}(\frac{\partial \vec{j}}{\partial t} + \vec{\nabla} \cdot (\vec{v}\vec{j} + \vec{j}\vec{v}) + \frac{1}{\Omega \tau_{ei}} + \frac{1}{\Omega \tau_{en}})B\vec{j} + \vec{j} \times \vec{B}$$

$$+ \frac{f^2 \Omega \tau_{in}}{B}\left[\vec{\nabla}\vec{P}_e \times \vec{B} - \left(\vec{j} \times \vec{B}\right) \times \vec{B}\right]$$

$$\vec{E}_0 = \vec{E} + \vec{v} \times \vec{B}$$

$\Omega = \dfrac{eB}{m_e}$ electron gyration frequency

If ignore electron inertia & pressure gradient:

$$\sigma\vec{E}_0 = \vec{j} + \frac{\sigma}{n_e e}\vec{j}\times\vec{B} - \frac{\sigma f^2 \Omega \tau_{in}}{n_e e B}(\vec{j}\times\vec{B})\times\vec{B}$$

$$\sigma = \frac{n_e e^2 m_e^{-1}}{\tau_{ei}^{-1} + \tau_{en}^{-1}}$$ electron conductivity at B = 0

if

$$\vec{j} \,//\, \vec{B}: \quad \sigma_0 E_0 = j$$

$$\vec{j} \perp \vec{B}: \quad \sigma_3 \vec{E}_0 = \vec{j} + \frac{\sigma_3}{n_e e}\vec{j}\times\vec{B}$$

$$\sigma_3 = \frac{\sigma}{1 + f^2 B\Omega \frac{\tau_{in}}{n_e e}}$$ cowling conductivity

Solve above equation

$$\vec{j} = \sigma_1 \vec{E}_0 + \sigma_2 \frac{\vec{B}\times\vec{E}_0}{B}$$

$$\sigma_1 = \frac{\sigma_3}{1 + (\sigma_3 \frac{B}{n_e e})^2} \qquad \sigma_2 = \frac{\sigma_3 B}{n_e e}\sigma_1$$

If gas is fully ionized n_a=0

$$\sigma_1 = \frac{\sigma}{1 + \Omega^2 \tau_{ei}^2} \qquad \sigma_2 = \Omega \tau_{ei} \sigma_1 \qquad \sigma_3 = \sigma = \frac{n_e e^2 \tau_{ei}}{m_e}$$

Compare to

$$\vec{j} = \sigma\vec{E}$$

There is an additional term normal to $\vec{B}$ and $\vec{E}$, it is called Hall current

Induction Equation

Ohm's Law:

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times (-\vec{v} \times \vec{B} + \frac{\vec{j}}{\sigma}) \quad \Leftarrow \begin{cases} \vec{\nabla} \times \vec{B} = \mu \vec{j} \\ \vec{E} = \frac{\vec{j}}{\sigma} - \vec{v} \times \vec{B} \\ \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \end{cases}$$

$$= \nabla \times (\vec{v} \times \vec{B}) - \vec{\nabla} \times (\eta \vec{\nabla} \times \vec{B}) \quad \Leftarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla} \cdot \vec{\nabla} \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \eta \nabla^2 \vec{B} \qquad \eta = \frac{1}{\mu \sigma} \quad \text{magnetic diffusivity}$$

$$j = \frac{\vec{\nabla} \times \vec{B}}{\mu} \qquad \vec{E} = \vec{v} \times \vec{B} + \frac{\vec{j}}{\sigma}$$

For condition of the sun : $\vec{E} \approx \vec{v} \times \vec{B}$

A typical active region, B=100G, v=10^3 m/s, E_0=v_0 B_0=10 V/m

If no motion,

$$j = \frac{1}{\mu l} = 8 \times 10^{-4} \, Am^{-2} \qquad l = 10^7 \, m$$

$$\sigma = 10^3 \, ohmm^{-1}$$

$$E_0 = j_0 / \sigma = 8 \times 10^{-7} \, Vm^{-1}$$

Electric conductivity

Fully ionized collision- dominated plasma

$$\sigma = \frac{n_e e^2 \tau_{ei}}{m_e}$$

Collision time $\tau_{ei} = 0.266 \times 10^6 \frac{T^{3/2}}{n_e \ln \Lambda}$

Coulomb logarithm $\ln\Lambda$ is between 5 and 20

$$\sigma = 1.53 \times 10^{-2} \frac{T^{3/2}}{\ln \Lambda} \quad ohmm^{-1}$$

Diffusion constant $\eta = 5.2 \times 10^7 (\ln \Lambda) T^{-3/2} \quad m^2 s^{-1}$

typical value in solar chromosphere or corona : $\eta = 10^9 T^{-3/2}$

for turbulent plasma, anomalous collision time : $\tau^* = w_{pe}^{-1} \frac{nk_B T}{w}$

w_{pe} : plasma frequency

anomalous conductivity : $\sigma^* = \frac{n_e e^2 \tau^*}{m_e}$

Condition for this is electron conduction speed >thermal speed

Plasma equations

continuity equation : $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$

equation of motion : $\rho \dfrac{D\vec{v}}{Dt} = \rho(\dfrac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}) = -\nabla P + \vec{j} \times \vec{B} + \vec{F}$

$\dfrac{D}{Dt}$: derivative of following the motion

$\vec{j} \times \vec{B}$: Lorentz force

$\vec{F} = \vec{F}_g + \vec{F}_v$ viscosity ν : coefficient viscosity

$\vec{F}_v = \rho \nu \nabla^2 \vec{v}$

$\rho \nu = 2.21 \times 10^{-16} \dfrac{T^{5/2}}{\ln \Lambda} \quad kgm^{-1}s^{-1}$

$P = nkT \qquad n = n_p + n_e$

Perfect gas law

Energy equation:

or $$\rho T \frac{Ds}{Dt} = -\zeta$$

$$\rho \frac{De}{Dt} = -\frac{p}{\rho}\frac{D\rho}{Dt} = -\zeta$$

e: internal energy, s: entropy

Ideal gas:

$$e = c_\upsilon T$$

$$c_\rho = c_\upsilon + \frac{K}{m} \qquad \gamma = \frac{c_\rho}{c_\upsilon} = \frac{N+2}{N}$$

N: # of degrees of freedom

Different terms for ζ

- Thermal conduction:

$$\vec{q} = -\vec{K}\nabla T$$

$\vec{K}$ thermal conduction tensor

- Radiation:

$$l_r = \nabla \cdot q_r = -K_r \nabla^2 T$$

K_r heat diffusivity

optical thick, this is valid;

optical thin, $l_r = n_e n_H Q(T)$

- Heating:

$$H = \rho\varepsilon + H_r + H_w$$

nuclear + viscous disipatation + waves

- Energetics:

$$E \cdot j = \frac{j^2}{\sigma} + \vec{v} \cdot \vec{j} \times \vec{B}$$

A few dimension less parameters

Reynolds number : $R_e = \dfrac{l_0 V_0}{\gamma}$

Magnetic Reynolds number : $R_m = \dfrac{l_0 V_0}{\eta}$ coupling between flow and magnetic fields

Solar atmosphere : $R_m >> 1$ coupling is strong

Mach number : $M = \dfrac{V_0}{C_s}$ $C_s = \sqrt{\dfrac{r p_0}{\rho_0}}$ speed of sound wave

Alfven mach number : $M_A = \dfrac{V_0}{V_A}$ $V_A = \dfrac{\beta_0}{(\mu \rho_0)^{1/2}}$ Alfven speed

plasma β : $\beta = \dfrac{2 \mu p_0}{B_0^{\ 2}}$ $\dfrac{\text{Gas}}{\text{Magnetic}}$ pressure

$\beta << 1$ low β magnetic dominant - corona

possby number : $R_0 = \dfrac{V_0}{l_0 \Omega}$ $\dfrac{\text{inertial}}{\text{Coriolis}}$

Prandtl number : $p_m = \dfrac{R_m}{R_e} = \dfrac{\gamma}{\eta}$ $\dfrac{\text{Viscoucity}}{\text{diffusion}}$

Example. Sunspot

$$l_0 = 10^7\,\text{m} \quad v_0 = 10^3\,\text{m/s} \quad \Omega = 10^{-6}\,\text{s}^{-1}$$

$$T_0 = 10^4\,\text{k} \quad n_0 = 10^{20}\,\text{m}^{-3} \quad \beta_0 = 10^3\,\text{G}$$

$$R_m = 3\times 10^7, \quad \text{C}_\text{s} = 2\times 10^4\,\text{m/s}\,(M = 0.05)$$

$$V_A = 3\times 10^5\,\text{m/s} \quad R_e = 10^{12} \quad R_0 = 100$$

More about Induction Equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B}) + \eta\nabla^2 B$$

• if $R_m = \dfrac{l_0 V_0}{\eta} << 1$ 1st term can be dropped, $\dfrac{\partial \vec{B}}{\partial t} = \eta\nabla^2 B$ →diffusion equation

diffusion time scale

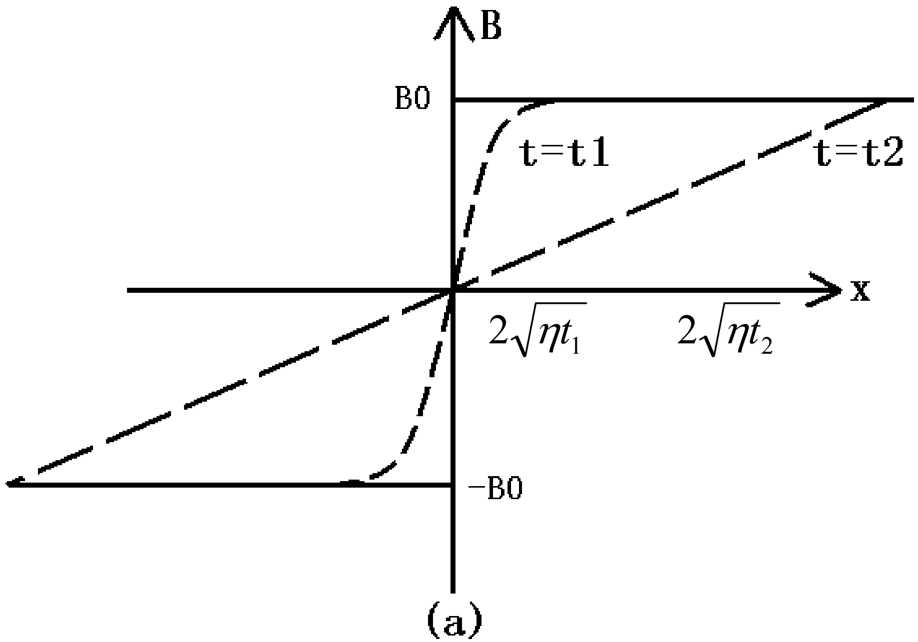
$$\tau_d = \frac{l_0^{\;2}}{\eta} = 1.9\times 10^{-8}\,\frac{l_0^{\;2} T^{\frac{3}{2}}}{l_n \Lambda}$$

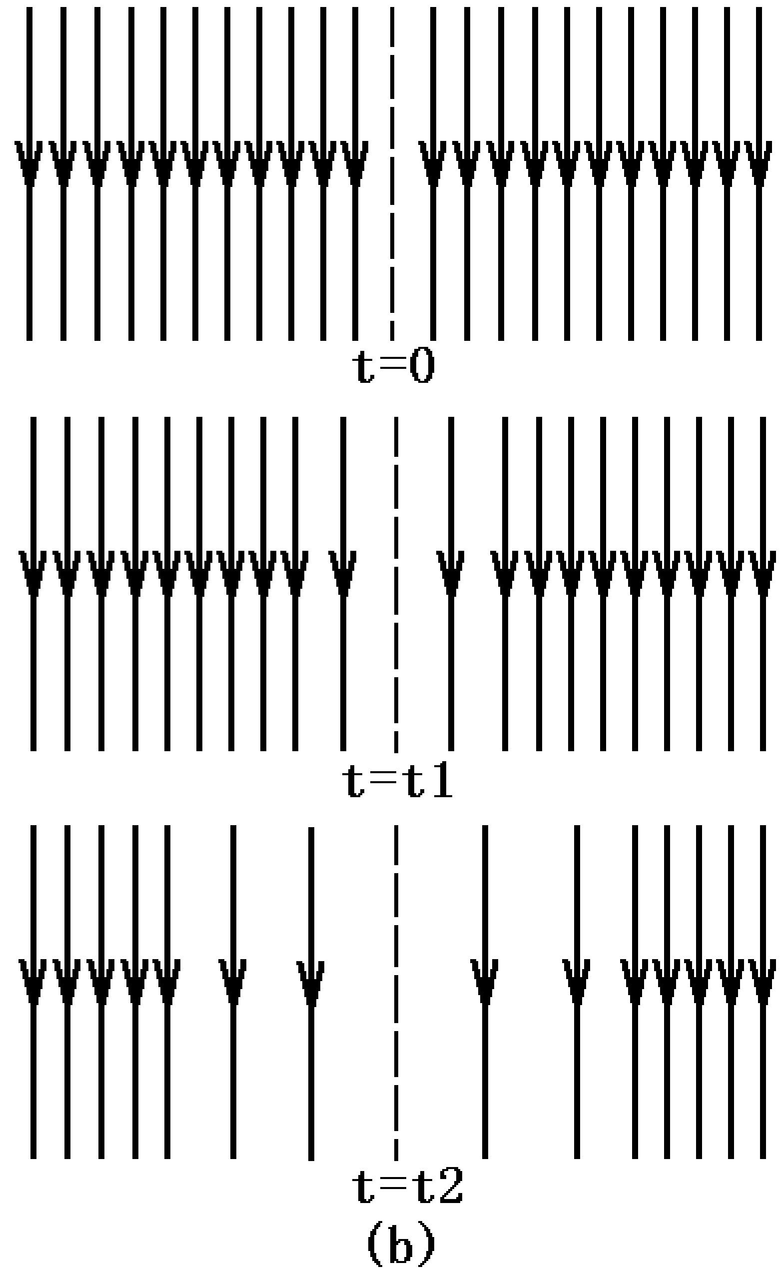
$$T = 10^6\,\text{K} \quad \eta = 10^{15}\,\text{m}^{-3} \quad l_0 = 10^7\,\text{m} \quad \tau_\text{d} = 10^{14}\,\text{S}$$

For solar flare $\tau_\text{d} = 100 \text{ to } 1000\text{S}$ $\quad l_0 = 100 \text{ to } 1000\,\text{m}$

• if $R_m >> 1$ frozen-in condition, $\vec{B}$ Moves with flow -- generally true in solar condition

prove: $\dfrac{D}{\partial t}\left(\dfrac{\vec{B}}{\rho}\right) = \left(\dfrac{\vec{B}}{\rho}\cdot\nabla\right)\vec{v}$ is similar to fluid motion $\dfrac{\text{D}\rho}{\text{Dt}} = -\rho\nabla\cdot v$





A diffusing current sheet:

(a): the variation with time of the magnetic field strength;

(b): a sketch of the magnetic field lines at three times

Lorentz Force

$$F_L = \vec{j} \times \vec{B} = \frac{1}{\mu}(\nabla \times \vec{B}) \times \vec{B} = (\vec{B} \cdot \vec{\nabla})\frac{\vec{B}}{\mu} - \nabla(\frac{B^2}{2\mu})$$

magnetic tention magnetic pressure

Example (1) uniform field

$$\vec{j} = \frac{D \times \vec{B}}{\mu} = 0 \quad \vec{F}_L = 0$$

(2) $\vec{B} = B_0 e \times \vec{y}$ $\quad \vec{j} = \frac{B_0 e}{\mu} \times \vec{z}$ $\quad \vec{j} \times \vec{B} = -\frac{B_0{}^2 e^{2x}}{\mu}\vec{x}$

pressure term

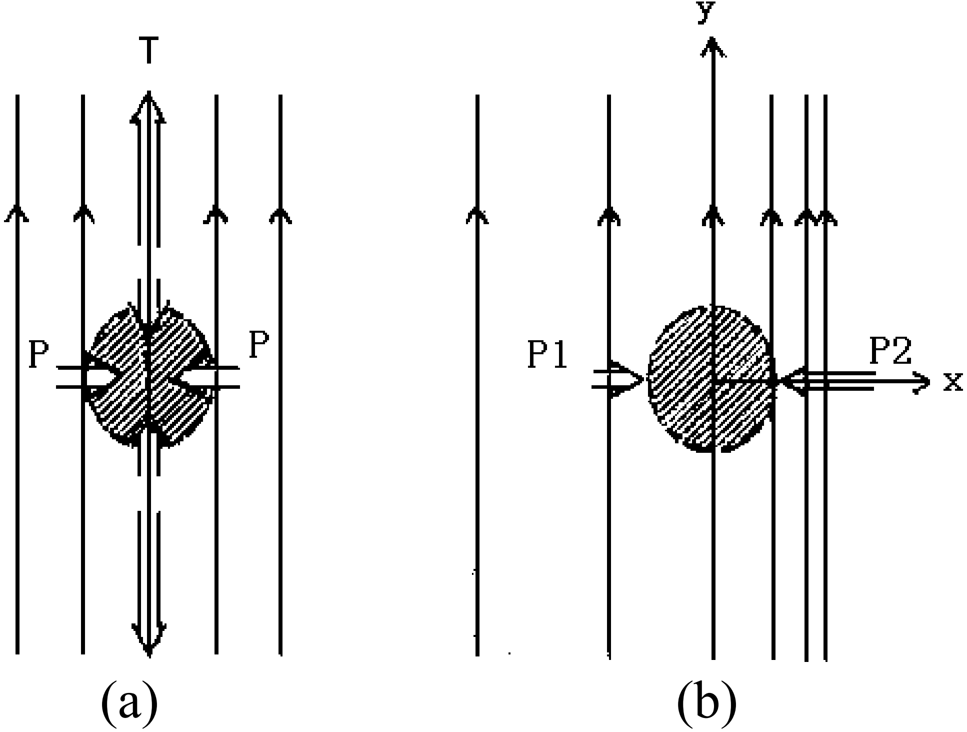
(3) $\vec{B} = -y\hat{x} + \hat{y}$ $\quad \vec{j} = \frac{\hat{z}}{\mu}$

$(\vec{j} \times \vec{B})_{y=0} = -\frac{\hat{x}}{\mu}$ tension term

(4) X-type neutral point

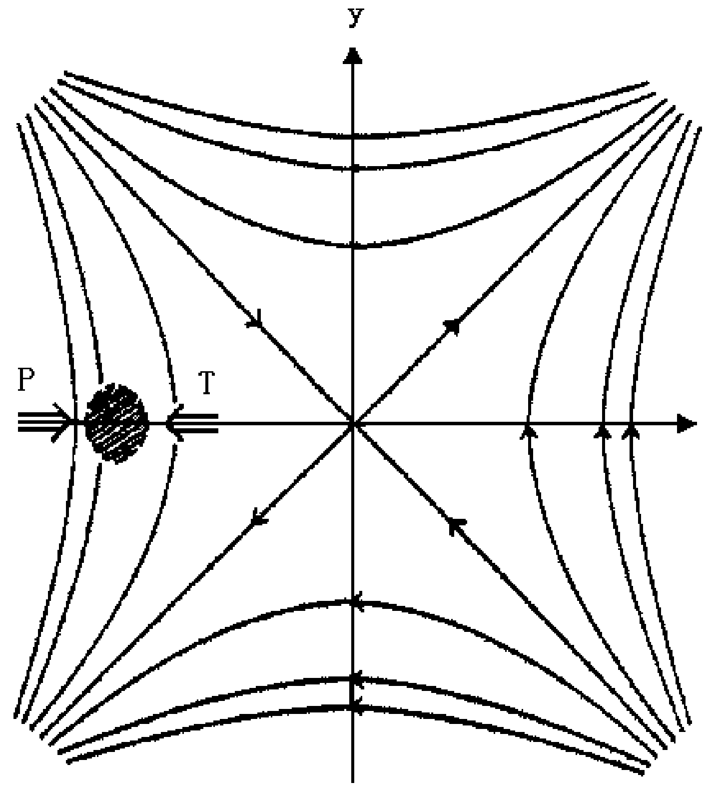
$B_0 = y\hat{x} + x\hat{y}$ (solution $y^2 - x^2 = \mathrm{const}$)

T balances with P, but unstable.

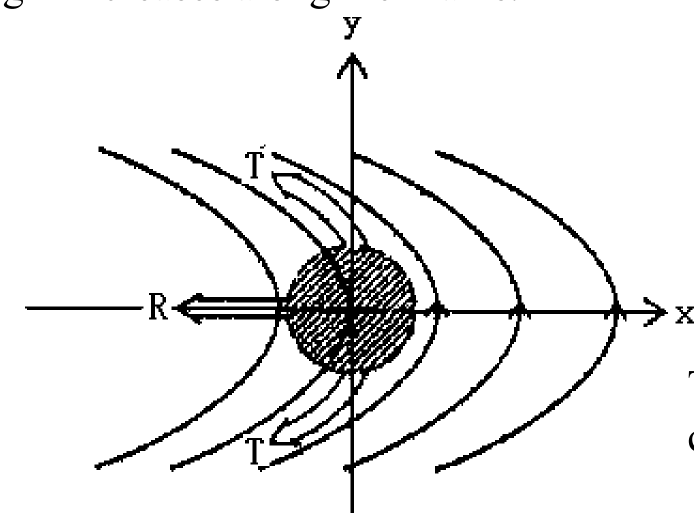


The magnetic pressure P and tension T forces due to:

(a) A uniform field; (b) a unidirectional field whose strength increases along the x-axis.



The magnetic field lines near an X-type neutral point in equilibrium with no current



The resultant magnetic force (R) due to a symmetrically curved field (Equation 2.59)

Flux tubes and current sheets

Flux tubes

$$F = \int B \cdot ds = \text{constant}$$

$$\mathrm{P} + \frac{\mathrm{B}^2}{2\mu} = \text{constant} \quad \text{(dark sunspots)}$$

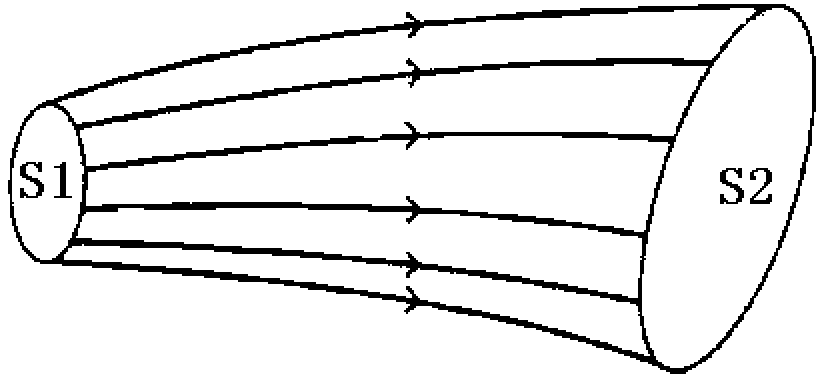
current sheet:

B varies a lot in a very short spatical scale L

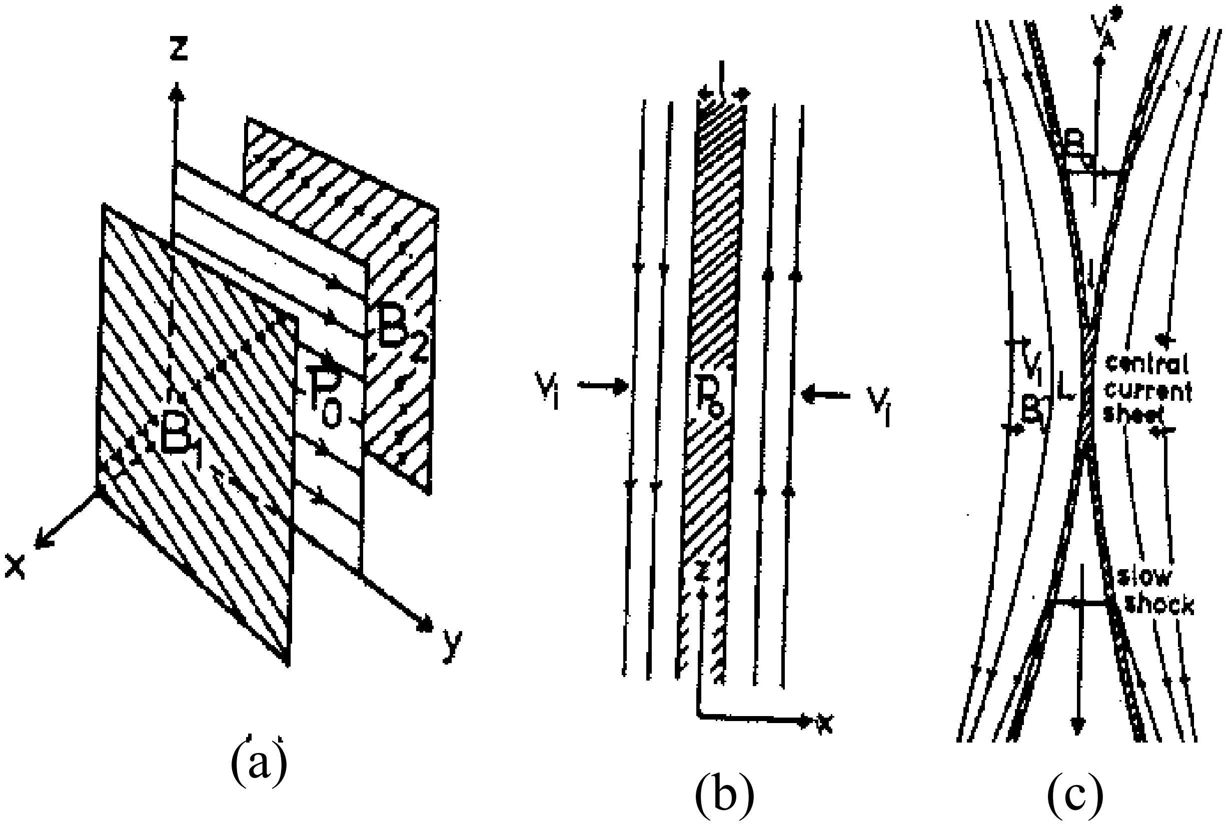
current density $j = \dfrac{B}{\mu L}$

if $\mathrm{v} = 0$, current sheet diffuses at a speed of η / l, η: magnetic diffusivity

enhanced plasma pressure in the center of the sheet expels material away at a speed of $\mathrm{V_A}$, reconnection



A section of a magnetic flux tube



(a): a current sheet in the yz plane across which the magnetic field rotates from B1 to B2;

(b): a section across a neutral current sheet in the centre of which the magnetic field vanishes and the plasma pressure is P0;

©: the reconnection of magnetic field lines by their passage through a current sheet. The central sheet bifurcates into two pairs of slow shocks.

Homework

1. Using reasonable parameters in solar photosphere, chromosphere, corona and in sunspots, calculate the following parameters: Alfven speed, sound speed, and plasma ß.

2. Refer to Equation 2.3, assume a reasonable diffusion constant, and magnetic field strength, compute magnetic topology as a function of time using the induction equation (ignore the velocity term).