Part 1. Particle Kinetics of Thermal Plasmas

1.1 Debye Shielding, Collective Behavior of Plasma, Plasma and Cyclotron Oscillations

Debye length:

$$\lambda_D \equiv \left[\frac{kT}{4\pi ne^2}\right] = 690\text{cm}\left(\frac{T[10^4\text{K}]}{n[\text{cm}^{-3}]}\right)^{1/2}$$

Debye number:

$$N_D = n\frac{4\pi}{3}\lambda_D^3 = 1.38\times 10^9\frac{(T[10^4\text{K}])^{3/2}}{(n[\text{cm}^{-3}])^{1/2}}$$

Since $N_D \gg 1$, we expect collective behaviors.
Plasma frequency:

$$\omega_p = \sqrt{\frac{4\pi e^2 n}{m_e}} = 5.64\times 10^4\text{Hz}(n[\text{cm}^{-3}])^{1/2}$$

Plasma	n [cm^{-3}]	ω_p
Interstellar Medium	1	100 kHz
Earth Ionosphere	10^4	10 MHz (radio)
Fusion experiment	10^{15}	100 GHz (mm)
Sun's core	10^{25}	10^{17} Hz (SX)

Cyclotron frequency:

$$\omega_c = \frac{eB}{m_e c} = 1.76\times 10^7\text{Hz}B[\text{gauss}]$$

Plasma	B [gauss]	ω_c
Interstellar Medium	$\sim 10^{-6}$	$\sim$ 100 kHz
Earth/Sun magnetosphere	~ 1	$\sim$ 10 MHz (radio)
Neutron star magnetosphere	10^{12}	$\sim 10^{19}$ GHz (HX)
Fusion experiment	10^6	$\sim 10^{13}$ Hz (IR)

1.2 Coulomb Collisions of a "Test" Particles with "Field" Particles

It is essential to understand basic physics of Coulomb scattering (or collisions) in this study as Coulomb scattering will be the dominant physical process in a plasma. For an order-of-magnitude estimate, we take

* Note from the lecturer (JL): I suppose Ph 777 is primary for solar MHD study. During its first three weeks we will review some fundamental aspects of the plasma physics as appropriate for the rest of the course. Last week, we learned Debye shielding, and plasma/cyclotron frequencies as well as the motions of individual particles moving in a largescale electromagnetic fields. Here I continue some related issues that we can do with thermal (or just mean, if not Maxwellian,) velocity distribution. In the next (week 3), I will intend to introduce the kinetic theory for warm plasmas. Most of materials here are derived from the lecture note of Ph136 at Caltech which I attended in 1989-90. The credit of this material goes to Dr. Roger Blandford.

Test particle	Field particle
m	m_f
v	v_f
$-e$	Ze

and assume $m \ll m_f$ and $(1/2)mv^2 \gg (1/2)m_f v_f^2$ (i.e., $v \gg v_f$).

We begin with a single Coulomb collision as shown below:

There is a critical impact parameter (which gives $(1/2)mv^2 = Ze^2/b_0$)

$$b_0 \equiv \frac{2Ze^2}{mv^2}$$

by which we distinguish two regimes of the deflection angle.

For $b \gg b_0$, deflected through a very small angle in which case we use $mv\theta_D = \int Ze^2 b/(b^2+v^2t^2)^{3/2}dt = 2Ze^2/vb$. Thus

$$\theta_D = \frac{b}{b_0}.$$

For $b \le b_0$, the deflection angle is of order $\pi/2$.

$$\theta_D \sim \frac{\pi}{2}.$$

The test particle's energy loss $-\triangle E$ equals to the energy gained by the field particle $\frac{1}{2}m_f(\triangle v_f)^2$. Also the momentum conserves, $mv\theta_D = m_f \triangle v_f$, so

$$\triangle E = -\frac{1}{2}m_f(\triangle v_f)^2 = -\frac{m}{m_f}\left(\frac{b_0}{b}\right)^2 E.$$

Next we turn from an individual Coulomb collisions to the net, statistically averaged effect of many collisions, and consider the mean time t_D required for the orbit of the field particles to get deflected by $\pi/2$ from initial direction.

If single large-angle scattering is dominant source of this deflection, t_D would be:

$$t_D = \frac{1}{n_f \sigma v} = \frac{1}{n_f \pi b_0^2 v} \qquad \text{(single large angle scattering)}$$

However, the cumulative, random walk effects of many single scattering actually produces a net $\pi/2$ deflection in a time scale shorter than this. In this case the mean deflection angle will vanish, but the mean square deflection angle will be:

$$\langle\Theta^2\rangle = \int_{b_{min}}^{b_{max}} \left[\frac{b_0}{b}\right] n_f v t 2\pi b db = n_f 2\pi b_0^2 v t \ln\left[\frac{b_{max}}{b_{min}}\right].$$

In this case, the deflection time (when $\langle\Theta^2\rangle \sim 1$) is therefore

$$t_D = \frac{1}{n_f 2\pi b_0^2 v \ln\lambda} \qquad \text{(multiple random scattering).}$$

This is shorter by $(1/2)\ln\Lambda$ than the t_D for single large-angle scattering.

In addition to t_D, we want to know the mean time for the energy of the test particle to change significantly. Since

$$\frac{\triangle E}{E} = -\frac{m}{m_f}\left(\frac{b_0}{b}\right)^2 = -\frac{m}{m_f}\theta_D^2,$$

by analogy,

$$\langle\frac{\triangle E}{E}\rangle = -\frac{m}{m_f}\langle\Theta_D\rangle^2.$$

Therefore,

$$t_E = \frac{m_f}{m} t_D.$$

If we know Λ, we will be able to calculate t_D, t_E, ... etc. It can immediately be seen that $b_{max} = \lambda_D$. The minimum impact parameter b_{min} has different values depending on whether quantum mechanical (QM) wave packet spreading is important or not.

Suppose the particle is in a QM wave packet with transverse size l_0 as it nears the field particle. Then it will have a QM spread $\triangle p_y \simeq \hbar/l_0$ in its initial transverse momentum. After time τ the packet's size will be

$$l^2(\tau) = l_0^2 + \left[\frac{\triangle p_y}{m}\tau\right]^2 \simeq l_0^2 + \left[\frac{\hbar\tau}{m l_0}\right]^2 \geq \frac{\hbar\tau}{m}$$

$$l \geq \sqrt{\frac{\hbar\tau}{m}} \sim \sqrt{\frac{\hbar b}{m v}}$$

where τ is a time for its encounter with the field particle. Since wave packet can be no smaller than $\sqrt{\hbar b/mv}$, the nearest distance that the test particle can come to the field particles is $b_{\text{nearest}} = \hbar/mv$, the deBroglie wavelength. If $b_{nearest} < b_0$, the large-angle scatterings are allowed and $b_{min} = b_0$. If $b_{nearest} > b_0$, the large-angle scatterings are prevented from occurring by wave packet spreading, $b_{min} = b_{nearest}$. In summary,

$$b_{min} = \max\,[b_0, \lambda_{DeBroglie}] = \max\,[\frac{2Ze^2}{mv^2}, \frac{\hbar}{mv}]$$

$$\text{and} \qquad \Lambda = \lambda_D / b_{min}$$

Note that we can express the ratio of the two b_{min}'s as

$$\frac{\hbar/mv}{b_0} = \frac{v/c}{2Z\alpha}$$

where $\alpha \equiv e^2/\hbar c \simeq 1/137$, the fine structure constant. This means that (for the field particles of either protons or electrons i.e. $Z = 1$) wave packet spreading is important, and $b_{min} = \hbar/mv$ at test particle speeds $> 2\alpha c$; scattering is unimportant and $b_{min} = b_0$ at speeds $< 2\alpha c$.

If the test particle is an electron ($m = m_e$) and has a velocity given by $(1/2)mv^2 = (3/2)kT$ and if plasma is made of hydrogen ($Z = 1$), the switch of b_{min} from $\hbar/mv$ to b_0 occurs at $\frac{1}{2}mv^2 = \frac{3}{2}kT = 2\alpha^2 mc^2$, i.e., at $kT \simeq$ (one Rydberg), i.e. $T \simeq 1 \times 10^4$ K. By contrast, if the test particle is proton then the switch occurs at $kT \simeq (m_p/m_e)\times$(one Rydberg), i.e. $T \simeq 2 \times 10^7$ K. See the table (from Spitzer 1962) in HW1.

In some literature (e.g., Melrose 1986)

$$\begin{aligned} \ln \Lambda_C &\simeq 16.0 - \frac{1}{2}\ln n_e + \frac{3}{2}\ln T_e & T_e \leq 7 \times 10^4 \text{ K} \\ &\simeq 21.6 - \frac{1}{2}\ln n_e + \ln T_e & T_e \geq 7 \times 10^4 \text{ K.} \end{aligned}$$

1.3 Thermal Equilibration Time Scales in a Plasma

The time required for particle of one species with another achieving a Maxwellian distribution would be similar to the time for significant energy exchange, i.e., $t_{eq} \sim t_E$.

(1) $t_{eq,\,e-e} \equiv$ time required for electrons to equilibrate with each other. Assume electrons begin with typical individual energies of order T_e where T_e is the temperature to which they are going to equilibrate, but their initial velocity distribution is non-Maxwellian. Choose a typical electron as *test* particle and all other electrons as the *field* particles.

$$m = m_e, \qquad m_f = m_e, \qquad mv^2 \sim m_f v^2 \sim 3kT_e$$

$$n_f = n, \quad b_0^2 = \left(\frac{2e^2}{mv^2}\right)^2 = \left(\frac{2e^2}{3kT_e}\right)^2$$

$$t_{eq,\,e-e} = \left(\frac{m_f}{m}\right)\frac{1}{n_f 2\pi b_0^2 v \ln\Lambda} = \frac{3^{3/2}}{8\pi}\frac{m_e^{1/2}(kT_e)^{3/2}}{e^4 n \ln\Lambda}$$

A more careful calculation based on the Fokker-Planck equation gives $3^{3/2}/8\pi = 0.21$ instead of $3/4\sqrt{\pi} = 0.42$.

$$t_{eq,\ e-e} = \frac{3}{4\sqrt{\pi}} \frac{m_e^{1/2}(kT_e)^{3/2}}{e^4 n \ln\Lambda} = \left[\frac{1.6\times 10^9\mathrm{s}}{(\ln\Lambda)/10}\right]\left[\frac{(kT_e/1\mathrm{keV})^{3/2}}{n/1\mathrm{cm}^{-3}}\right]$$

(2) $t_{eq,\ i-i} \equiv$ time required for ions to equilibrate with each other. We replace m_e by m_p and T_e by T_i

$$t_{eq,\ i-i} = \left[\frac{m_p}{m_e}\right]^{1/2}\left[\frac{T_i}{T_e}\right]^{3/2} t_{eq,\ e-e} \approx 43 t_{eq,\ e-e} = \left[\frac{7\times 10^{10}\mathrm{s}}{(\ln\Lambda)/10}\right]\left[\frac{(kT_e/1\mathrm{keV})^{3/2}}{n/1\mathrm{cm}^{-3}}\right]$$

(3) $t_{eq,\ i-i} \equiv$ time required for electrons to equilibrate with ions. Set $m = m_e$, $m_f = m_p$ and $T_e = T_i$. Correspondingly $mv^2 = m_f v_f^2 = 3kT_e$. We then find $t_{eq,\ e-i} = (m_p/m_e)t_{eq,\ e-e}$, but a more careful calculation based on Fokker-Planck equation gives an additional factor, $1/2^{3/2} = 0.35$:

$$t_{eq,\ e-i} = \frac{1}{2^{3/2}}\left[\frac{m_p}{m_e}\right] t_{eq,\ e-e} \approx 650 t_{eq,\ e-e} = \left[\frac{1\times 10^{12}\mathrm{s}}{(\ln\Lambda)/10}\right]\left[\frac{(kT_e/1\mathrm{keV})^{3/2}}{n/1\mathrm{cm}^{-3}}\right]$$

Note that the energy equilibration times t_{eq} are almost always enormously long compared to $1/\omega_p$, $1/\omega_c$, and other dynamical time scales of a plasma. As a result, it is common that the electron and ion velocity distributions are not Maxwellian and the non-Maxwellian nature plays an important role in the dynamics and evolution of the plasma. In analyzing the dynamics, one must use the phase space distribution function $N(x, v, t)$ for electrons and ions, respectively.

1.4 Thermoelectric Transport Coefficients

Electrons are much more mobile than ions, and are mainly responsible for the transport of heat and charge through a plasma. But there typically are two impediments to an electron's motion: Coulomb collisions and the Lorentz force $\mathbf{v}\times\mathbf{B}$ due to magnetic field $\mathbf{B}$.

If $t_{D,e} \ll 1/\omega_c$ then magnetic field has little effect on thermal and electrical conductivity. But much more common is $t_{D,e} \gg 1/\omega_c$ in which an electron typically makes many circuits around the magnetic field line before Coulomb scattering significantly affects its orbit. In this case the electrons have great difficulty traveling distances larger than their Lamor radii

$$r_L \equiv \frac{\nu}{\omega_c} \simeq \frac{3kTm_ec^2}{eB} = 131\ \mathrm{cm}\ \sqrt{\frac{kT[\mathrm{keV}]}{B[\mathrm{gauss}]}}$$

in directions transverse to the field. This suppression, strictly speaking, makes MHD approach in valid. However, when MHD predicts negligible current flow across field lines, it recovers its validity.

Here we shall derive the expression for the electrical conductivity κ_e and thermal conductivity κ_T in regimes where the principal impediment to electron mobility is Coulomb scattering:

In the presence of a temperature gradient ∇T, not only will a flow of heat, $\mathbf{Q}$, appear, but an electric current $\mathbf{J}$ will also flow. Similarly an electric produces a flow of heat. In the absence of magnetic fields,

$$\mathbf{Q} = -\kappa_T \nabla T - \beta \mathbf{E}, \qquad \mathbf{J} = \kappa_e \mathbf{E} + \alpha \nabla T$$

where α and β are called *thermoelectric transport coefficients.* The derivation can be found in some books (e.g., Kittel (1958) *"Elementary Statistical Physics"*). Here we will use the above result to do an order of magnitude estimate.

(a) In a situation where $\nabla T = 0$ we will show

$$\kappa_e \sim \frac{ne^2 t_{D,e}}{m_e}, \qquad \beta \sim \frac{kT}{e}\kappa_e.$$

Calculate $J = -nev$. To estimate v from $F\delta t = m_e v$ with δt taken as the mean diffusion time, $t_{D,e}$,

$$v = -\frac{eE}{m_e} t_{D,e}$$

and

$$J = \frac{ne^2 t_{D,e}}{m_e} E. \qquad \text{hence} \qquad \kappa_e = \frac{ne^2 t_{D,e}}{m_e}$$

Also

$$Q = -nvE = n\frac{eE}{m_e} t_{D,e}(kT) = \frac{kT}{e}\Big(\frac{ne^2 t_{D,e}}{m_e}\Big)E = \frac{kT}{e}\kappa_e E$$

Thus

$$\beta \sim \frac{kT}{e}\kappa_e.$$

(b) In a situation where $\mathbf{E} = 0$, we will show

$$\kappa_T \sim kn\frac{kT_e}{m_e} t_{D,e}, \qquad \alpha \sim \frac{e}{kT_e}\kappa_T$$

Consider the temperature and density gradient exist such that a region with (T, n, v) is separated from a region with $(T + \delta T, n + \delta n, v')$ by a distance $l = \lambda_e$, the mean free path of electrons. Then

$$Q = n_e v(kT) - (n + \delta n)v' k(T + \delta T) = nk \cdot k\sqrt{\frac{kT}{m_e}}\delta T$$

if we use $\delta T = l \cdot (dT/dx)$ and $v\ t_{D,e} \equiv l$ we have

$$Q = nk\frac{kT}{m_e} t_{D,e} \nabla T$$

Also, we have $J = -nev = -nev^2 v^{-1} \sim ne(kT/m_e)(t_{D,e}/l)$. Since $J = \alpha \nabla T$, $\alpha \sim (e/kT_e)\kappa_T$.

(c) According to the thermodynamics these coefficients are not independent but satisfy a specific relation (the Onsager relation):

$$\beta = \alpha T_e + \frac{5}{2}\frac{kT_e}{e}\kappa_e$$

Our order of magnitude estimate shows that

$$\frac{\beta}{\kappa_e}\frac{\alpha}{\kappa_T} \sim \frac{kT}{e}\frac{e}{kT} = O(1)$$

This is consistent with the relation, $1 - \beta\alpha/\kappa_e\kappa_T = 0.419$ (see below). Also, $\beta \sim (kT/e)\kappa$ and $\alpha \sim kn(e/m_e)t_{D,e} \simeq k/e\kappa_e \rightarrow \alpha T \sim (kT)/e\kappa_e$. This is consistent with the Onsager relation.

(d) If a temperature gradient persists for sufficiently long, the effective thermal conductivity reduces to about 42% of that in the absence of $\mathbf{E}$. To see it, consider that after build-up of charge separation,

$$J = 0 \rightarrow E = -\frac{\alpha}{\kappa_e}\nabla T.$$

Therefore

$$q = \Big(\frac{\beta\alpha}{\kappa_e} - \kappa_T\Big)\nabla T = -\kappa_{T,eff}\nabla T$$

which defines $\kappa_{T,eff}/\kappa_T = 1 - \beta\alpha/\kappa_e\kappa_T$. Spitzer showed that for Hydrogen plasma,

$$\frac{\alpha\beta}{\kappa_e\kappa_T} = 0.581.$$

1.5 Anomalous Resistivity

If Coulomb collision in a thermal plasma is the only process, electrical conductivity (called Spitzer value) will be

$$\sigma = \frac{ne^2}{m\nu_{th}} \quad \text{with} \quad \nu_{th} = \frac{4\pi ne^4 \ln\Lambda}{m^2 v_{th}^3}$$

where ν_{th} is the thermal collision frequency.

However, due to collective excitation, electromagnetic fields may build up, which is more effective than Coulomb scattering at deflecting the orbits of individual electrons and ions and exchanging energies with them. Correspondingly, the deflections by the excitation's fields cause the transport coefficients κ_e, κ_T, α, and β to be much lower than predicted by Coulomb scattering. i.e. they produce larger resistivity to the flow of current and heat. The enhanced resistivities are called *anomalous*. Enhanced resistivity increases the collision rate and can make the plasma approach the equilibrium more quickly.

For instance, low-frequency ion-sound turbulence leads to anomalous conductivity given as:

$$\sigma^* \equiv \frac{n_e e^2}{m_e \nu^*} \quad \text{with} \quad \nu^* \approx \omega_{pe}\Big(\frac{W}{nkT}\Big).$$

1.6 Dreicer Field

In the context of electron acceleration by electric field, there is a concept of "run-aways." The idea is based on the fact that the acceleration in an electric field is independent of the speed of the electron (only depends on charge and electric field E) whereas the slowing down due to collisions $\sim v^{-3}$ (see above and HW2). So there is a critical velocity (or energy) above which electrons will freely run away unhindered by collisions. To find the relation between the critical velocity v_{cr} and the electric field, we start from the balance of the electric field force with thermal collisions:

$$eE = mv_d \nu_{th}$$

where v_d is a drift (away from equilibrium) velocity due to imposition of an external field $\mathbf{E}$.

The electric field is regarded "weak" when $v_d \ll v_{th}$ so that the plasma is not greatly displaced *en masse*. If we increase the electric field to make $v_d = v_{th}$ so that virtually all the electrons run away. This is called Dreicer field and thus given by

$$E_D = \frac{mv_{th}}{e}\nu_{th} = \frac{4\pi e^3}{m}\ln\Lambda\left(\frac{n}{v_{th}^2}\right)$$

In other words, in a weak electric field ($E < E_D$) supra-thermal electrons can run away. Since $E \sim v^{-2}$, we can express v_{cr} as

$$v_{cr} = \frac{v_{th}}{\sqrt{E/E_D}}$$

Note: 1. Due to runaways the tail of the Maxwellian distribution becomes depleted and collisions feed electrons into the depleted tail and there will be continuous supply of electrons. 2. You may express $E_D = 4\pi e/\lambda_D^2$ if it seems more meaningful to you.

1.7 Charged Particle Motions in a Largescale Electromagnetic Fields

– done in the last class.

References

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Jackson, J.D. (1962) Classical Electrodynamics
Kittel, C. (1958) Elementary Statistical Physics
Spitzer, L. Jr. (1962) Physics of Fully Ionized Gases
Melrose, D. B. (1986) Instabilities in space and laboratory plasmas
Reif, F. Fundamentals of Statistical and Thermal Physics

Homework (week 2)

HW1 : Spitzer (1962) gives values of $\ln \Lambda$ as:

Reproduce this table using the formulae derived in this class, noticing where the expression for b_{min} switches. Take electron as test particle, and electron and positron as field particles. Compare the result with the expression given by Priest. Which value of $\ln \Lambda$ would be appropriate for the solar corona?

HW2: Derive the expression for Spitzer's electrical conductivity σ and thermal collision frequency ν_{th} when Coulomb collisions dominate (see sec. 1.5). Discuss why $\nu_{th} \propto v_{th}^{-3}$.