

# Final Exam Solution May 11, 2005

1) a)  $y'' + ey' = 1 + y$

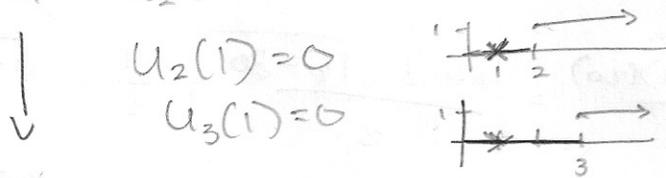
All terms are linear in  $y$ .

$y'' \rightarrow$  second order

b)  $g(t) = \alpha + u_2(t) - 3\alpha u_3(t)$

$g(1) = 2$

$2 = \alpha + u_2(1) - 3\alpha u_3(1)$



$\alpha = 2$

c)  $f(x) = (e^x + e^{-x}) \sin(x)$

$f(-x) = (e^{-x} + e^x) \sin(-x)$

$= (e^x + e^{-x})(-\sin(x))$

$= -(e^x + e^{-x}) \sin(x) = -f(x)$

$f(-x) = -f(x) \therefore \underline{f(x) \text{ is odd}}$

2. a)  $y' + 2y = e^{-2x}, y(0) = 1$

$p(x) = 2, g(x) = e^{-2x}$

$\mu(x) = e^{\int p(x) dx} = e^{\int 2 dx} = e^{2x}$

$y = \frac{\int \mu(x) g(x) dx + c}{\mu(x)} = \frac{\int e^{2x} e^{-2x} dx + c}{e^{2x}} = \frac{\int dx + c}{e^{2x}}$

$$y = \frac{\int dx + c}{e^{2x}} = \frac{x+c}{e^{2x}} = xe^{-2x} + ce^{-2x}$$

$$y(0) = 1 = (0)e^{-2(0)} + ce^{-2(0)} = \underline{c = 1}$$

$$\boxed{y = (x+1)e^{-2x}}$$

$$b) y'' - 4y' + 20y = 0 \quad y(0) = 0 \quad y'(0) = 8$$

$$r^2 - 4r + 20 = 0$$

$$r = \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(20)}}{2} = \frac{+4 \pm \sqrt{16 - 80}}{2} = \frac{4}{2} \pm \frac{\sqrt{-64}}{2} = 2 \pm \frac{8i}{2} = 2 \pm 4i$$

$$\alpha = 2 \quad \beta = 4$$

$$y = e^{\alpha t} [c_1 \cos(\beta t) + c_2 \sin(\beta t)]$$

$$y = e^{2t} (c_1 \cos(4t) + c_2 \sin(4t))$$

$$y' = e^{2t} \cdot 2(c_1 \cos(4t) + c_2 \sin(4t)) + e^{2t} (-c_1 \cdot 4 \sin(4t) + c_2 \cdot 4 \cos(4t))$$

$$y' = e^{2t} [2c_1 \cos(4t) + 2c_2 \sin(4t) - 4c_1 \sin(4t) + 4c_2 \cos(4t)]$$

$$y(0) = 0 = e^{2(0)} (c_1 \cos(0) + c_2 \sin(0)) = \underline{c_1 = 0}$$

$$y'(0) = 8 = e^{2(0)} [2(0) \cos(0) + 2c_2 \sin(0) - 4(0) \sin(0) + 4c_2 \cos(0)]$$

$$8 = 4c_2 \quad \underline{c_2 = 2}$$

$$\boxed{y(t) = e^{2t} [2 \sin(4t)]}$$

$$3. \quad a) \quad 2y' - y = \frac{1}{y}$$

$$2 \frac{dy}{dt} = y + \frac{1}{y} = \frac{y^2+1}{y}$$

$$2dy \cdot \frac{y}{y^2+1} = dt$$

$$\int \frac{2y}{y^2+1} dy = \int dt$$

$$\ln|y^2+1| = t + c$$

$$e^{\ln|y^2+1|} = ce^t$$

$$y^2+1 = ce^t$$

$$y^2 = ce^t - 1$$

$$y = \pm \sqrt{ce^t - 1}$$

$$y(x) = \frac{\int x e^x dx + c}{\frac{x}{e^x}}$$

$$y(x) = \frac{-x e^{-x} - e^{-x} + c}{x e^x}$$

$$= -1 - \frac{1}{x} + \frac{c e^x}{x}$$

$$y(x) = \frac{c e^x - 1}{x} - 1$$

$$b) \quad y' + \frac{y}{x} = 1 + y$$

$$y' + \frac{y}{x} - y = 1$$

$$y' + y \left( \frac{1}{x} - 1 \right) = 1$$

$$\mu = e^{\int p(x) dx}$$

$$\int p(x) dx = \int \frac{1}{x} - 1 dx$$

$$= \ln(x) - x$$

$$\mu = e^{\ln x - x} = e^{\ln x} / e^x = \frac{x}{e^x}$$

$$y(x) = \frac{\int \mu(x) q(x) dx + c}{\mu(x)}$$

$$y(x) = \frac{\int \frac{x}{e^x} \cdot 1 dx + c}{\frac{x}{e^x}}$$

$$u = x \quad du = dx$$

$$dv = e^{-x} dx \quad v = -e^{-x}$$

$$\int x e^{-x} dx = -x e^{-x} - \int -e^{-x} dx$$

$$-x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}$$

$$4. a) x' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} x \quad x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} x = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} x$$

$$\det \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-1-\lambda) - (2)(3) = 0$$

$$(\lambda-4)(\lambda+1) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 - 6 = 0$$

$$\lambda^2 - 3\lambda - 10 = 0 = (\lambda-5)(\lambda+2)$$

Eigenvalues

$$\lambda_1 = 5, \lambda_2 = -2$$

$$\lambda_1 = 5$$

$$5 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$5x_1 = 4x_1 + 2x_2$$

$$x_1 = 2x_2$$

$$\frac{x_1}{x_2} = \frac{2}{1}$$

$$\xi_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvectors

$$\lambda_2 = -2:$$

$$-2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-2x_1 = 4x_1 + 2x_2$$

$$-6x_1 = 2x_2$$

$$\frac{x_1}{x_2} = -\frac{1}{3}$$

$$\xi_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\vec{x}(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$x(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2c_1 e^{5(0)} \\ c_1 e^{5(0)} \end{pmatrix} + \begin{pmatrix} -c_2 e^{-2(0)} \\ 3c_2 e^{-2(0)} \end{pmatrix} \Rightarrow$$

$$2 = 2c_1 - c_2$$

$$1 = c_1 + 3c_2$$

$$2 = 2c_1 - c_2$$

$$2 = 2c_1 + 6c_2$$

$$0 = -7c_2$$

$$c_2 = 0$$

$$1 = c_1 + 3(0)$$

$$c_1 = 1$$

$$\vec{x}(t) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

4. b.  $y'' = 2 \quad y(0) = 1 \quad y'(0) = 0$

$$s^2 Y - sy(0) - y'(0) = \frac{2}{s}$$

$$s^2 Y - s = \frac{2}{s}$$

$$s^2 Y = \frac{2}{s} + s$$

$$Y = \frac{2}{s^3} + \frac{1}{s}$$

$$\mathcal{L}^{-1}\left(Y = \frac{2}{s^3} + \frac{1}{s}\right) \Rightarrow \boxed{y(t) = t^2 + 1}$$

5. a)  $y'' - 2y' + y = \frac{e^t}{t}$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1$$

$$y_1 = e^t$$

$$y_2 = te^t$$

$$\underline{y(t) = C_1 e^t + C_2 te^t}$$

Gen. solution

for corresponding

homogeneous solution

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = \underline{e^{2t}}$$

$$Y(t) = \left(-\int \frac{y_2 g}{W} dt\right) y_1 + \left(\int \frac{y_1 g}{W} dt\right) y_2$$

$$Y(t) = \left(-\int \frac{te^t \cdot \frac{e^t}{t}}{e^{2t}} dt\right) e^t + \left(\int \frac{e^t \cdot \frac{e^t}{t}}{e^{2t}} dt\right) te^t$$

$$Y(t) = \left(-\int dt\right) e^t + \left(\int \frac{dt}{t}\right) te^t$$

$$Y(t) = -te^t + \ln(t) \cdot te^t$$

Particular solution

$$y(t) = C_1 e^t + C_2 te^t - te^t + te^t \ln(t) \quad \leftarrow \text{absorbed into } C_2 \text{ term}$$

$$\boxed{y(t) = C_1 e^t + C_2 te^t + te^t \ln(t)}$$

Complete  
solution

$$b. 2y'' - 7y' + 3y = e^t$$

$$Y = Ae^t$$

$$2r^2 - 7r + 3 = 0$$

$$Y' = Ae^t$$

$$(2r-1)(r-3) = 0$$

$$Y'' = Ae^t$$

$$r = 1/2, 3$$

$$2Ae^t - 7Ae^t + 3Ae^t = e^t$$

$$y_1 = e^{t/2}$$

$$y(t) = c_1 e^{t/2} + c_2 e^{3t}$$

$$-2Ae^t = e^t \quad -2A = 1$$

$$y_2 = e^{3t}$$

$$A = -1/2$$

$$Y(t) = -\frac{1}{2} e^t$$

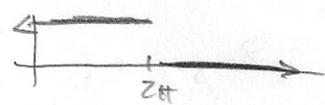
$$y(t) = c_1 e^{t/2} + c_2 e^{3t} - \frac{1}{2} e^t$$

$$b. a) y'' + y = g(t) \rightarrow g(t) = \begin{cases} 1 & 0 \leq t \leq 2\pi \\ 0 & t > 2\pi \end{cases}$$

$$y(0) = 0$$

$$y'' + y = 1 - u_{2\pi}(t)$$

$$y'(0) = 0$$



$$\mathcal{L}[y'' + y] = \mathcal{L}[1 - u_{2\pi}(t)] \quad g(t) = 1 - u_{2\pi}(t)$$

$$s^2 Y - sy(0) - y'(0) + Y = \frac{1}{s} - \frac{e^{-2\pi s}}{s}$$

$$(s^2 + 1)Y = \frac{1 - e^{-2\pi s}}{s}$$

$$Y = \frac{1 - e^{-2\pi s}}{s(s^2 + 1)} = \frac{1}{s(s^2 + 1)} - \frac{1}{(s^2 + 1)s} e^{-2\pi s}$$

$$\frac{1}{(s^2 + 1)s} = \frac{As + B}{s^2 + 1} + \frac{C}{s} = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$1 = As^2 + Bs + Cs + C$$

$$A + C = 0 \quad B = 0 \quad C = 1$$

$$A = -1$$

$$Y = \frac{1}{s} - \frac{s}{s^2 + 1} - \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-2\pi s}$$

$$F(s) = \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

$$f(t) = 1 - \cos(t)$$

$$y(t) = 1 - \cos(t) - u_{2\pi}(t)(1 - \cos(t - 2\pi))$$

$$y(t) = 1 - \cos(t) - u_{2\pi}(t)(1 - \cos(t))$$

$$b) y(\pi) = 1 - \cos(\pi) - u_{2\pi}(\pi)(1 - \cos(\pi))$$

$$= 1 - (-1) - 0(1 - \cos(\pi))$$

$$= 2$$

$$c) \lim_{t \rightarrow \infty} y(t) \Rightarrow 1 - \cos(t) - u_{2\pi}(t)(1 - \cos(t))$$

$$\lim_{t \rightarrow \infty} u_{2\pi}(t) = 1$$

$$1 - \cos t - 1 + \cos t$$

$$= 0$$

$$\boxed{\lim_{t \rightarrow \infty} y(t) = 1 - 0 = 1}$$

$$7. x^3 y'' + xy' - y = 0$$

$$x^3(v''x + 2v') + x(v'x + v) - xv = 0$$

$$v''x^4 + 2v'x^3 + v'x^2 + vx - vx = 0$$

$$v''x^4 + v'(2x^3 + x^2) = 0$$

$$v'' + v' \left( \frac{2x^3 + x^2}{x^4} \right) = 0$$

$$v'' + v' \left( \frac{2}{x} + \frac{1}{x^2} \right) = 0$$

$$\mu = e^{\int \left( \frac{2}{x} + \frac{1}{x^2} \right) dx} = e^{2 \ln x - \frac{1}{x}} = x^2 e^{-\frac{1}{x}}$$

$$v' = \frac{\int 0 \cdot x^2 e^{-\frac{1}{x}} dx + c}{x^2 e^{-\frac{1}{x}}} = \frac{e^{-\frac{1}{x}}}{x^2}$$

$$W = \begin{vmatrix} x & -e^{\frac{1}{x}} \\ 1 & \frac{1}{x^2} e^{\frac{1}{x}} \end{vmatrix} = \frac{x}{x^2} e^{\frac{1}{x}} + e^{\frac{1}{x}} = \left( \frac{1}{x} + 1 \right) e^{\frac{1}{x}}$$

$$e^{\frac{1}{x}} \neq 0 \quad \forall x$$

$$\frac{1}{x} + 1 = 0 \quad \text{at } x = -1$$

$$y_1 = x \quad * y_2 = v(x) y_1 = xv$$

$$y_1' = 1 \quad y_2' = v'(x) y_1 + v(x) y_1'$$

$$y_1'' = 0 \quad y_2'' = v''(x) y_1 + v'(x) y_1' + v'(x) y_1' + v(x) y_1''$$

$$= v'' y_1 + 2v' y_1' + v y_1''$$

$$= v'' x + 2v'(1) + v(0)$$

$$* y_2'' = v'' x + 2v'$$

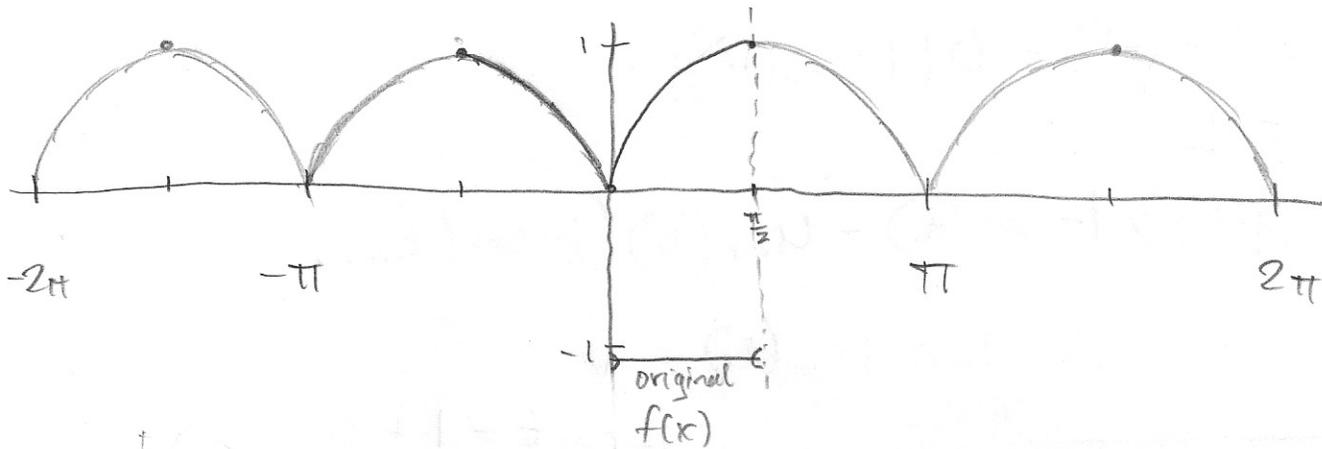
$$* y_2' = v' x + v(1) = v' x + v$$

$$v = \int \frac{e^{\frac{1}{x}}}{x^2} dx = - \int e^u du = -e^u = -e^{\frac{1}{x}}$$

$$u = \frac{1}{x} \quad du = -\frac{1}{x^2} dx$$

$$\boxed{y_2 = v y_1 = -e^{\frac{1}{x}} x}$$

8. a)  $f(x) = \sin(x)$   $0 \leq x < \pi/2$



b)  $f(x) = 1$   $0 < x < 1$

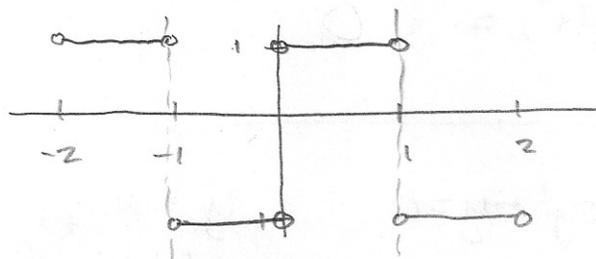
Fourier sine:

$$F(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$$

Periodically  
extended  
 $f(x)$

$$T=2$$

$$L = \frac{T}{2} = \frac{2}{2} = 1$$



$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$f(x)$  is odd and  $\sin$  is odd

$\therefore f(x) \sin\left(\frac{m\pi x}{L}\right)$  is even

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= \frac{2}{1} \int_0^1 1 \cdot \sin\left(\frac{m\pi x}{1}\right) dx$$

$$= 2 \int_0^1 \sin(m\pi x) dx$$

$$= \frac{-2}{m\pi} \cos(m\pi x) \Big|_0^1 = \frac{-2}{m\pi} [\cos(m\pi) - \cos(0)]$$

$$b_m = \frac{-2}{m\pi} ((-1)^m - 1)$$

$$\cos(m\pi) = (-1)^m$$

$$F(x) = \sum_{m=1}^{\infty} \frac{-2}{m\pi} [(-1)^m - 1] \sin\left(\frac{m\pi x}{L}\right)$$