# Bit error rate in multipath wireless channels with several specular paths

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In this letter a recursive and computationally efficient new formula for bit error rate (BER) in multipath channels with several specular components is derived. Using Jensen's inequality it is shown that one specular component, Rice fading, provides the lowest possible BER. Using Lagrange multipliers to solve a constrained optimization problem, it is further shown that for two specular components, BER is maximum when the two components have equal strengths. BERs for three and four specular paths are studied and compared numerically. The results show the impact of the number of specular paths on BER, as well as the smallest and largest BERs.

*Introduction:* Bit error rate (BER) of digital modulations has been extensively studied for multipath channels and in the presence of diffuse components (Rayleigh fading), as well as diffuse plus one specular component (Rice fading) [1]. The probability density function (pdf) of the signal envelope for more than one specular component is discussed in [2]-[4]. In this letter new results are derived using Jensen's inequality and Lagrange multipliers for constrained optimization, to determine the impact of the number of specular components on BER, and the highest and lowest values of BER.

*Problem formulation:* The received signal in a multipath fading channel with $N$ specular components and a diffuse component can be represented by $R\exp(j\Theta)=\sum_{i=1}^{N}a_i\exp(j\Phi_i)+A_0\exp(j\Phi_0)$ [2] [4], where $R$ and $\Theta$ are the received amplitude and phase, $j^2=-1$, $a_i$ and $\Phi_i$ are the amplitude and phase of the $i$-th specular component, and $A_0$ and $\Phi_0$ are the amplitude and phase of the diffuse component. The phases $\Phi_i$ , $i=0,1,2,...,N$ are independent and identically distributed uniform random variables over $[0,2\pi)$, whereas $a_i$'s are constant and $A_0$ has a Rayleigh distribution with average power $\Omega_0 = E[A_0^2]$ [2]. Let us define the superposition of specular components as $B_N \exp(j\Psi_N) = \sum_{i=1}^{N} a_i \exp(j\Phi_i)$. Conditioned on $B_N$, the pdf of $R$ is the Rice pdf [4]

$$f_{R|B_N}(r \mid b_N) = (2\Omega_0^{-1} r)\exp(-\Omega_0^{-1}(r^2 + b_N^2))\, I_0(2\Omega_0^{-1} b_N r), \tag{1}$$

where $I_0(.)$ is the zero-th order modified Bessel function. Let $U = R^2$ and $V_N = B_N^2$, which upon substitution into (1) result in the following unconditional pdf for $U$

$$f_U(u) = E_{V_N}[\Omega_0^{-1} \exp(-\Omega_0^{-1}(u + V_N)) I_0(2\Omega_0^{-1}\sqrt{V_N u})], \tag{2}$$

where $E$ is expectation. The BER expression of several modulations in additive white Gaussian noise is an exponential function of $U$ [1]. Here we consider binary differential phase shift keying whose BER, conditioned on $U$, is given by $P(U) = 0.5\exp(-\gamma_b U)$, where $\gamma_b$ is the signal to noise ratio (SNR) per bit. Using (2), average BER with $N$ specular components, $\bar{P}_N = \int_0^{\infty} P(u) f_U(u) du$, can be written as

$$\bar{P}_N = E_{V_N}\left[\frac{1}{2\Omega_0}\int_0^{\infty} \exp\left(-\gamma_b u - \frac{u+V_N}{\Omega_0}\right) I_0\left(\frac{2}{\Omega_0}\sqrt{V_N u}\right) du\right] = E_{V_N}\left[\frac{1}{2(\gamma_b\Omega_0 + 1)} \exp\left(\frac{-\gamma_b V_N}{\gamma_b \Omega_0 + 1}\right)\right], \tag{3}$$

where the integral in (3) is solved using Mathematica.

*Lowest possible BER:* Because $\gamma_b(\gamma_b\Omega_0 + 1)^{-1} > 0$, the term $\exp(-\gamma_b(\gamma_b\Omega_0 + 1)^{-1} V_N)$ in (3) is a convex function in $V_N$. Therefore, Jensen's inequality [5] results in

$$\bar{P}_N \geq 0.5(\gamma_b\Omega_0 + 1)^{-1} \exp\left(-\gamma_b(\gamma_b\Omega_0 + 1)^{-1}\Omega_N\right), \tag{4}$$

where $V_N$ in the exponential in (3) is replaced by $E[V_N] = \Omega_N = \sum_{i=1}^{N} a_i^2$, to obtain the right-hand side of the inequality in (4). The lower bound in (4) holds for any $N$, and can be achieved for $N = 1$. This is because according to eq. (8.213) in [1], BER in Rice fading is exactly the same as the right-hand side of (4). This completes our proof that for a fixed total power of specular components $\Omega_N$, Rice fading results in the lowest possible BER.

*Recursive formula for BER:* Without loss of generality and to simplify the notation, let $\Phi_1 = 0$. For $N = 2$, the sum of two vectors (specular components) results in $B_2 \exp(j\Psi_2) = a_1 + a_2 \exp(j\Phi_2)$. Using the cosine formula in the triangle with sides $B_2$, $a_1$ and $a_2$, we obtain $B_2^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos\Phi_2$. With $V_2 = B_2^2$ and $V_1 = a_1^2$, this can be written as $V_2 = V_1 + a_2^2 + 2a_2\sqrt{V_1}\cos\Phi_2$. When the third vector $a_3 \exp(j\Phi_3)$ is added, we can similarly write $V_3 = V_2 + a_3^2 + 2a_3\sqrt{V_2}\cos(\Phi_3 - \Psi_2)$, conditioned on $V_2$ and $\Psi_2$. By adding vectors one by one we obtain the recursion $V_i = V_{i-1} + a_i^2 + 2a_i\sqrt{V_{i-1}}\cos(\Phi_i - \Psi_{i-1})$, $i = 2,3,...,N$, with $\Psi_1 = 0$ and conditioned on $V_{i-1}$.and $\Psi_{i-1}$. This approach is used in [6] to derive the amplitude pdf recursively, whereas we use it in a different way, to obtain new BER results, by computing the BER recursively. By substituting $V_N = V_{N-1} + a_N^2 + 2a_N\sqrt{V_{N-1}}\cos(\Phi_N - \Psi_{N-1})$ into (3) we obtain the following BER, conditioned on $V_{N-1}$, $\Psi_{N-1}$ and $\Phi_N$

$$P_N(V_{N-1}, \Psi_{N-1}, \Phi_N) = \frac{1}{2(\gamma_b\Omega_0 + 1)} \exp\left( -\frac{\gamma_b(V_{N-1} + a_N^2 + 2a_N\sqrt{V_{N-1}}\cos(\Phi_N - \Psi_{N-1}))}{\gamma_b\Omega_0 + 1} \right). \quad (5)$$

Based on the integral identity $I_0(x) = (2\pi)^{-1}\int_0^{2\pi} \exp(x\cos(\beta - \xi))d\beta$, eq. (5) can be averaged with respect to $\Phi_N$, which is uniformly distributed over $[0, 2\pi)$. Since the resulting expression depends only on $V_{N-1}$, the recursive average BER can be ultimately written as

$$\bar{P}_N = \frac{1}{2(\gamma_b\Omega_0 + 1)} E_{V_{N-1}}\left[ \exp\left( \frac{-\gamma_b}{\gamma_b\Omega_0 + 1}(V_{N-1} + a_N^2) \right) I_0\left( \frac{2\gamma_b a_N}{\gamma_b\Omega_0 + 1}\sqrt{V_{N-1}} \right) \right], \; N = 1,2,3,..., \quad (6)$$

where $V_0 \equiv 0$. Eq. (6) includes Rice BER, eq. (8.213) in [1], as a special case, for $N = 1$. When $N = 2$, since $V_1 = a_1^2$ is a constant, eq. (6) results in

$$\bar{P}_2 = 0.5(\gamma_b\Omega_0 + 1)^{-1} \exp\left( -\gamma_b(\gamma_b\Omega_0 + 1)^{-1}(a_1^2 + a_2^2) \right) I_0\left( 2\gamma_b(\gamma_b\Omega_0 + 1)^{-1} a_1 a_2 \right). \quad (7)$$

For $N = 3$ the expectation in (6) is over $V_2$, which depends on $\Phi_2$. Therefore

$$\bar{P}_3 = 0.5(\gamma_b\Omega_0 + 1)^{-1} E_{\Phi_2}\left[ \exp\left( -\gamma_b(\gamma_b\Omega_0 + 1)^{-1}(V_2 + a_3^2) \right) I_0\left( 2\gamma_b(\gamma_b\Omega_0 + 1)^{-1} a_3\sqrt{V_2} \right) \right]. \quad (8)$$

Following the same approach, eq. (6) for $N = 4$ ultimately results in

$$\bar{P}_4 = 0.5(\gamma_b\Omega_0 + 1)^{-1} E_{\Phi_3,\Phi_2}\left[ \exp\left( -\gamma_b(\gamma_b\Omega_0 + 1)^{-1}(V_3 + a_4^2) \right) I_0\left( 2\gamma_b(\gamma_b\Omega_0 + 1)^{-1} a_4\sqrt{V_3} \right) \right], \quad (9)$$

with $V_3$ defined before and $\sqrt{V_2}\cos(\Phi_3-\Psi_2)=(a_1+a_2\cos\Phi_2)\cos\Phi_3+a_2\sin\Phi_2\sin\Phi_3$. Equations (7)-(9) are computationally more efficient than integrals over products of Bessel functions (see [4] and references therein).

*Highest and lowest BERs using Lagrange multipliers:* The BER in (7) is a function of $a_1$ and $a_2$. For a fixed total specular power $a_1^2+a_2^2=\Omega_2$, BER varies as $a_1$ and $a_2$ change, and it is of interest to find out what values of $a_1$ and $a_2$ maximize or minimize the BER. This a constrained optimization problem, subject to the constraint $a_1^2+a_2^2=\Omega_2$. To solve this using Lagrange multipliers [7], we define the Lagrange function $L(a_1,a_2,\alpha)=\bar{P}_2+\alpha(a_1^2+a_2^2-\Omega_2)$, where $\alpha$ is the Lagrange multiplier. By setting partial derivatives of $L$ zero, we obtained three critical points. The points $(a_1,a_2)=(\sqrt{\Omega_2},0)$ and $(0,\sqrt{\Omega_2})$ indicate that the specular power is in one component only, which is Rice fading and is shown in (4) to be the lowest BER. The third critical point $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$ indicates two equal amplitude specular components. Using the Hessian matrix [7], we have shown the BER is maximum at this critical point.

*Numerical results:* Let $K=\Omega_N/\Omega_0$ be the specular to diffuse power ratio and $\mu_i=a_i^2/a_1^2$, $i=2,3,...,N$ be the $i$-th specular power ratio. Also consider unit total power, i.e., $\Omega_N+\Omega_0=1$, without loss of generality. In Fig. 1 the BER for $N=2$ is plotted versus $\mu_2$ and using eq. (7). In agreement with the theoretical derivation, BER is maximum when the two specular components have equal strength, $\mu_2=1$, and is minimum when one specular component is much stronger than the other, $\mu_2=0$ or $\infty$. The BER for $N=3$ is plotted in Fig. 2 versus $\mu_2$ and $\mu_3$, using eq. (8). Consistent with the analytical finding that minimum BER occurs when one specular component is much stronger than the others, we observe that the minima in Fig. 2 are located at $(\mu_2,\mu_3)=(0.01,0.01),(100,0.01),(0.01,100)$. The maxima appear to occur when $(\mu_2, \mu_3) = (100,100),(1,0.01),(0.01,1)$. This agrees with the constrained optimization result that maximum BER for two specular components happens when the amplitudes are equal. To study the impact of *N*, in Fig. 3 BER is plotted for equal-strength specular components, using (7)-(9). Rayleigh fading BER $0.5(\gamma_b + 1)^{-1}$ [1] is also plotted as a reference. For a fixed total specular power, as shown in (4) using Jensen's inequality, $N = 1$ (Rice fading) has the lowest BER. Moreover, $N = 2$ has the highest BER. As *N* increases, BER approaches the Rayleigh case.

*Conclusion:* Using a new BER formula derived in this letter for multipath channels with $N \geq 1$ specular paths, it is proved that $N = 1$, Rice fading, provides the lowest BER among all possible *N*'s. For equal-amplitude specular components, the results show that with a fixed total specular power, BERs lie between $N = 1$ and $N = 2$ curves, and $N = 4$ provides a BER higher than $N = 3$, especially at high SNRs.

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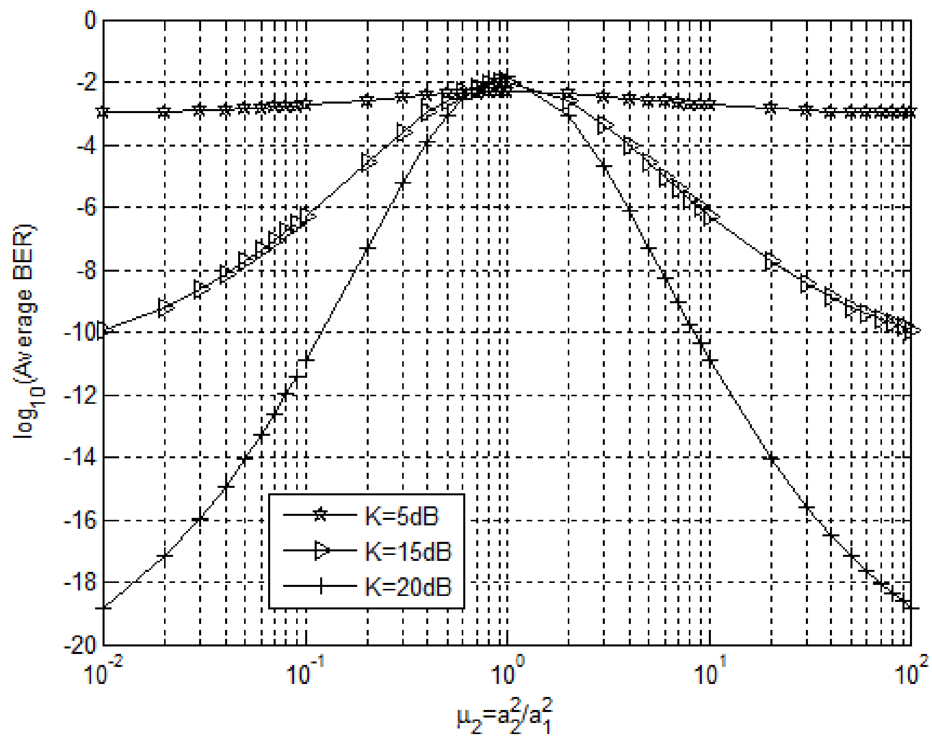
**Figure captions:**

Fig. 1 Bit error rate in a fading channel with two specular components versus the specular power ratio, with $\gamma_b = 20$ dB and different specular to diffuse power ratios $K$.

Fig. 2 Bit error rate in a fading channel with three specular components versus the specular power ratios, with $\gamma_b = 20$ dB and specular to diffuse power ratio $K = 20$ dB.

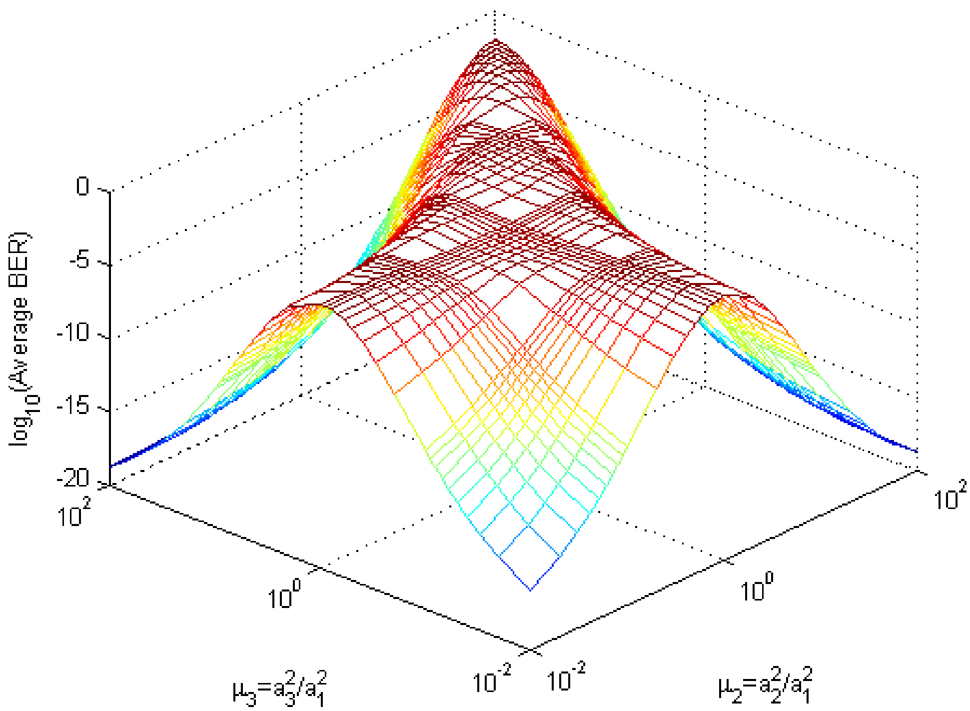
Fig. 3 Bit error rate in a fading channel with $N = 1,2,3,4$ specular components versus SNR, with equal amplitudes, $K = 20$ dB, and the corresponding Rayleigh bit error rate.

Figure 1



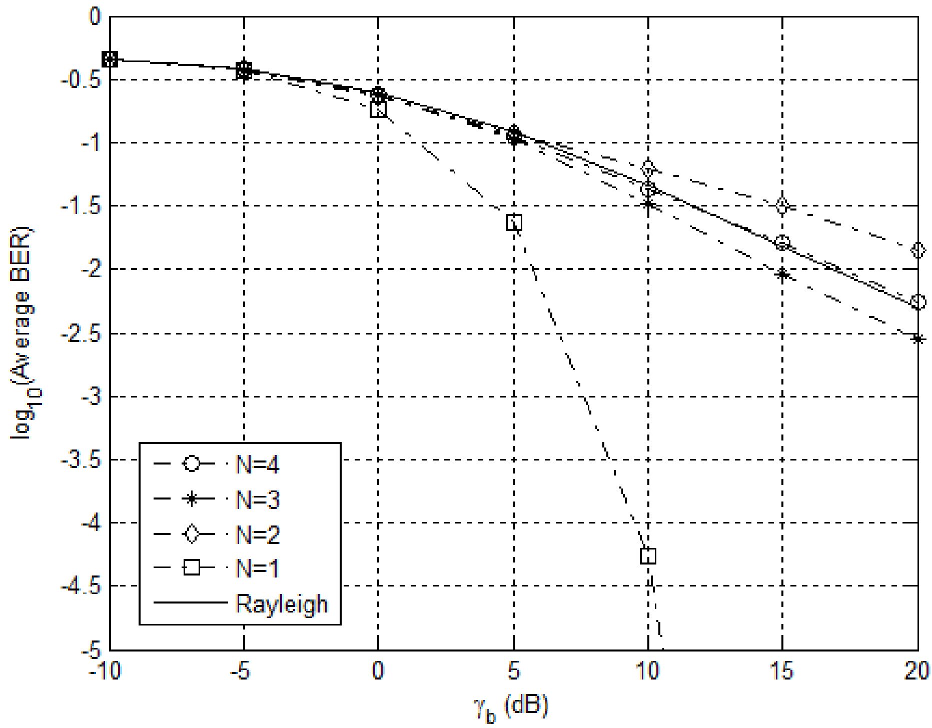
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
log10(Average BER)
10^-2
10^-1
10^0
10^1
10^2
μ2=a2^2/a1^2
K=5dB
K=15dB
K=20dB

Figure 2



log10(Average BER)
0
-5
-10
-15
-20
10^2
10^0
10^-2
μ3=a3^2/a1^2
10^-2
10^0
10^2
μ2=a2^2/a1^2

Figure 3



0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
-5
-10
-5
0
5
10
15
20
log10(Average BER)
γb (dB)
N=4
N=3
N=2
N=1
Rayleigh