Comparison of DPSK and MSK bit error rates for *K* and Rayleigh-lognormal fading distributions

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ABSTRACT

The composite Rayleigh-lognormal distribution is mathematically intractable for the analytical evaluation of such a communication system performance metric as bit error rate. The composite *K* distribution closely approximates the Rayleigh-lognormal and is potentially useful for analytical manipulations. In this contribution we derive the bit error rates of DPSK and MSK, in manageable closed forms, for the *K* distribution model of multipath fading and shadow fading, and show, numerically, the close agreement between these results and those based on the Rayleigh-lognormal distribution.

I. INTRODUCTION

The composite Rayleigh-lognormal (RL) distribution has been widely adopted for modeling the mixture of multipath fading and shadow fading, and measurements support this distribution for a number of wireless communication channels [1] [2]. Let *R* have a Rayleigh distribution with a random mode $\sqrt{Y}$. The probability density function of *R* conditioned on *Y* is $f_R(r|Y) = (r/Y)\exp(-r^2/2Y), \quad r \geq 0$. For mobile communication applications, the composite RL distribution was first introduced by Hansen and Meno assuming a lognormal distribution for *Y* [3], and simultaneously by Suzuki assuming a lognormal distribution for $\sqrt{Y}$ [1] [4]. However, the original idea apparently dates back to Sunde's work [5, p. 396]. This model has also been proposed independently for satellite communications [1].

The main drawback of the RL distribution is its complicated mathematical form [6, p. 44]. In fact, further manipulation of this distribution for the prediction of the average bit error rate (BER) for various modulation schemes, evaluation of the effect of different diversity methods, outage probability calculations, etc. is very difficult. On the other hand, *K* distribution, which approximates the RL distribution quite well [7], is promising for the above applications. In what follows, using the *K* fading model, we show that in contrast with the use of the RL fading model, we can obtain closed-form and easy-to-use expressions for the average BER of two basic modulation methods in wireless communications: differential phase shift keying (DPSK) and minimum shift keying (MSK).

II. BRIEF SUMMARY OF THE PROPERTIES OF THE *K* DISTRIBUTION

K distribution, a mixture of Rayleigh and gamma distributions, has been extensively used for modeling diverse scattering and propagation phenomena [8] [9] (and references therein). Assuming a gamma distribution with parameters α and β for *Y* [7]:

$$f_Y(y) = \frac{y^{\beta}}{(2\alpha^2)^{\beta+1}\Gamma(\beta+1)}\exp\left(\frac{-y}{2\alpha^2}\right), \quad y \geq 0, \tag{1}$$

with $\Gamma(.)$ as the gamma function, and then averaging $f_R(r|Y)$ with respect to *Y*, yields the *K* distribution:

$$f_R(r) = \int_0^\infty f_R(r|Y=y) f_Y(y)dy = \frac{2}{\alpha\Gamma(\beta+1)}\left(\frac{r}{2\alpha}\right)^{\beta+1} K_\beta\left(\frac{r}{\alpha}\right), \quad r \geq 0, \quad \alpha > 0, \quad \beta > -1. \tag{2}$$

In the above formula, $K_\beta(.)$ is the modified Bessel function of the second kind and order β. For the *K* distribution, α is the scale parameter, while β is the shape (or fading) parameter. The role of β in the

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[1] Of course, if *Y* is lognormally distributed, so is aY^b, where $a > 0$ and *b* are constants.

characterization of fading can be better understood by calculating the amount of fading *AF*, defined by $AF = Variance[R^2]/(E[R^2])^2$ [10] (also known as the "strength of intensity fluctuations" [8]). This can be done using the *k*th moment expression for the *K* distribution [7]:

$$E(R^k) = (2\alpha)^k \frac{\Gamma(1+k/2)\Gamma(\beta+1+k/2)}{\Gamma(\beta+1)}, \quad k = 0,1,2,\dots, \tag{3}$$

which in turn yields:

$$AF = \frac{\beta+3}{\beta+1}. \tag{4}$$

Note that $1 < AF < \infty$, as $-1 < \beta < \infty$. It can be shown that "$K \to$ Rayleigh" as $\beta \to \infty$, while "$K \to$ Dirac delta function at 0" as $\beta \to -1$.

Interestingly, the range of *AF* values for the *K* and RL distributions are the same. This can be seen by considering the lognormal distribution with parameters μ and λ for *Y* [7]:

$$f_Y(y) = \frac{1}{\sqrt{2\pi\lambda^2}\,y} \exp\left[\frac{-(\ln y - \mu)^2}{2\lambda^2}\right], \quad y \geq 0, \tag{5}$$

where ln is the natural logarithm. Using the *k*th moment of the RL distribution [7], we easily obtain $AF = 2\exp(\lambda^2) - 1$ which, similar to the *K* distribution, varies between 1 and ∞ for $0 < \lambda < \infty$.

III. Average BER of DPSK and MSK for *K* and RL Fading

Let $\gamma = E_b/N_0$, where E_b is the transmitted energy per bit and N_0 is the noise power spectral density. Then the BER for DPSK, conditioned on *Y*, is given by [11, eq. (5.72)]:

$$P_{b,DPSK}(Y) = \frac{1}{2(1+2\gamma Y)}. \tag{6}$$

Taking the expectation of (6) with respect to *Y* using (1) [12, eq. 3.383-10] leads us to the average BER of DPSK for *K* fading:

$$\overline{P}_{b,DPSK} = E\left(P_{b,DPSK}(Y)\right) = \frac{1}{2}\frac{1}{(4\alpha^2\gamma)^{\beta+1}} \exp\left(\frac{1}{4\alpha^2\gamma}\right)\Gamma\left(-\beta, \frac{1}{4\alpha^2\gamma}\right), \tag{7}$$

where $\Gamma(.,.)$ is the incomplete gamma function, i.e. $\Gamma(c,z) = \int_z^\infty e^{-t} t^{c-1} dt$.

The BER for MSK, conditioned on *Y*, is given by [11, eq. (5.28)]:

$$P_{b,MSK}(Y) = \frac{1}{2}\left[1 - \sqrt{\frac{2\gamma Y}{1+2\gamma Y}}\right]. \tag{8}$$

According to the definition of the Tricomi function $U(s,c,z) = \Gamma(s)^{-1}\int_0^\infty t^{s-1} e^{-zt}(1+t)^{c-s-1} dt$ [13, eq. 48:3:5] and after some algebric manipulations, taking the expectation of (8) with respect to *Y* using (1) results in the average BER of MSK for *K* fading:

$$\overline{P}_{b,MSK} = E\left(P_{b,MSK}(Y)\right) = \frac{1}{2}\left[1 - \frac{\Gamma(\beta+3/2)}{(4\alpha^2\gamma)^{\beta+1}\Gamma(\beta+1)} U\left(\beta+\frac{3}{2}, \beta+2, \frac{1}{4\alpha^2\gamma}\right)\right]. \tag{9}$$

For the RL fading, integral forms of the average BER are given in [14] for DPSK and BPSK (Note that BPSK and MSK have the same BER). These integrals are computed using the Gauss-Hermite integration method, the accuracy of which, for some parameter ranges, needs to be verified by the Simpson integration method [14]. Using approximate formulas for the BER, the average BER has been either derived in closed forms [3] [5, p. 396-397], or computed via numerical integration [15]. Only in [16], and for the simple BER expression of DPSK, an exact but complicated formula is derived for the average BER. These results strongly confirm the advantage of the *K* distribution over the RL distribution for average BER calculations.

IV. NUMERICAL RESULTS

In the dB scale, the assumption of lognormal distribution for shadow fading means that $10\log E(R^2|Y)$ is normally distributed with mean η_p dB and standard deviation σ dB, where log is the logarithm to the base 10 and subscript *p* indicates power. Based on the fact that $Y = E(R^2|Y)/2$, and using the relation $\log Y = \ln Y/\ln 10$ we obtain $10\log E(R^2|Y) = 10\log 2 + (10/\ln 10)\ln Y$. According to (5), $\ln Y$ is normally distributed with mean μ neper and standard deviation λ neper. Therefore $\sigma = (10/\ln 10)\lambda \approx 4.34\lambda$ dB, in agreement with [6, p. 37]. The same result can be obtained for $20\log E(R|Y)$ as a normal distribution with mean η_v dB and standard deviation σ dB, where subscript *v* represents voltage. Note that in contrast with the difference between η_p and η_v, σ is the same for lognormal variables $E(R^2|Y)$ and $E(R|Y)$ [2, pp. 220-221] [11, pp. 87-88].

In Figs. 1 and 2, average BERs are plotted for DPSK and MSK. For the *K* fading, (7) and (9) are used; while for the RL fading, (6) and (8) are averaged numerically with respect to *y* according to (5). The values $\sigma = 4.5, 8, 13$ dB in Figs. 1 and 2 correspond to urban areas, typical macrocells, and worst case of microcells, respectively [11, p. 88-89] (values equal to or less than 4.5 dB hold for land mobile satellite channel [16]). In neper unit we have $\lambda = 1.04, 1.84, 3$ (typical values for microwave amplitude scintillation over satellite links are $\lambda = 0.77, 1, 1.8$ [1]). The associated β values can be obtained by solving the equation $\lambda^2 = \Psi'(1+\beta)$ numerically, where $\Psi(.)$ is the psi function [7]. Hence $\beta = 0.35, -0.37, -0.65$ (note that β is inversely proportional to σ). Without a loss of generality we have taken $\alpha = 1$, which in turn yields $\mu = \ln 2 + \Psi(1+\beta)$ [7]. Inspection of Figs. 1 and 2 confirms the utility of the *K* distribution for BER prediction in multipath fading-shadow fading channels, instead of using the common RL distribution to model such a composite fading. The small discrepancies between the BER curves arise from the fact that the *K* and the RL distributions are not mathematically equal. For β close to -1 (large σ), the *K* distribution has a smaller peak [7, Fig. 1], and goes to zero faster. However, we can observe that these differences have only a small effect on the BER curves over a wide range of signal to noise ratios.

V. CONCLUSION

In this contribution we have shown how the *K* distribution provides closed-form BER expressions for DPSK and MSK, in terms of the tabulated special functions incomplete gamma and Tricomi (also available in Mathematica). Such BERs can be obtained only numerically for the commonly used Rayleigh-lognormal distribution model of multipath fading-shadow fading. The lognormal approximation for the Rayleigh-lognormal [2, p. 156-159] is valid only for large σ (discrepancies appear for small σ [2, p. 158]), and like the Rayleigh-lognormal does not lead to closed-form BER expressions (for a comparison of *K* and lognormal, see [9]). The key point that makes the *K* distribution preferable to the Rayleigh-lognormal distribution for analytic calculations, is the usage of the gamma distribution instead of the lognormal distribution for shadow fading. This alternative model has theoretical and experimental support [17]. It is anticipated that more compact results related to such issues as coverage, diversity, interference, and outage probability can be obtained, when the *K* fading model is used in place of the Rayleigh-lognormal fading model.

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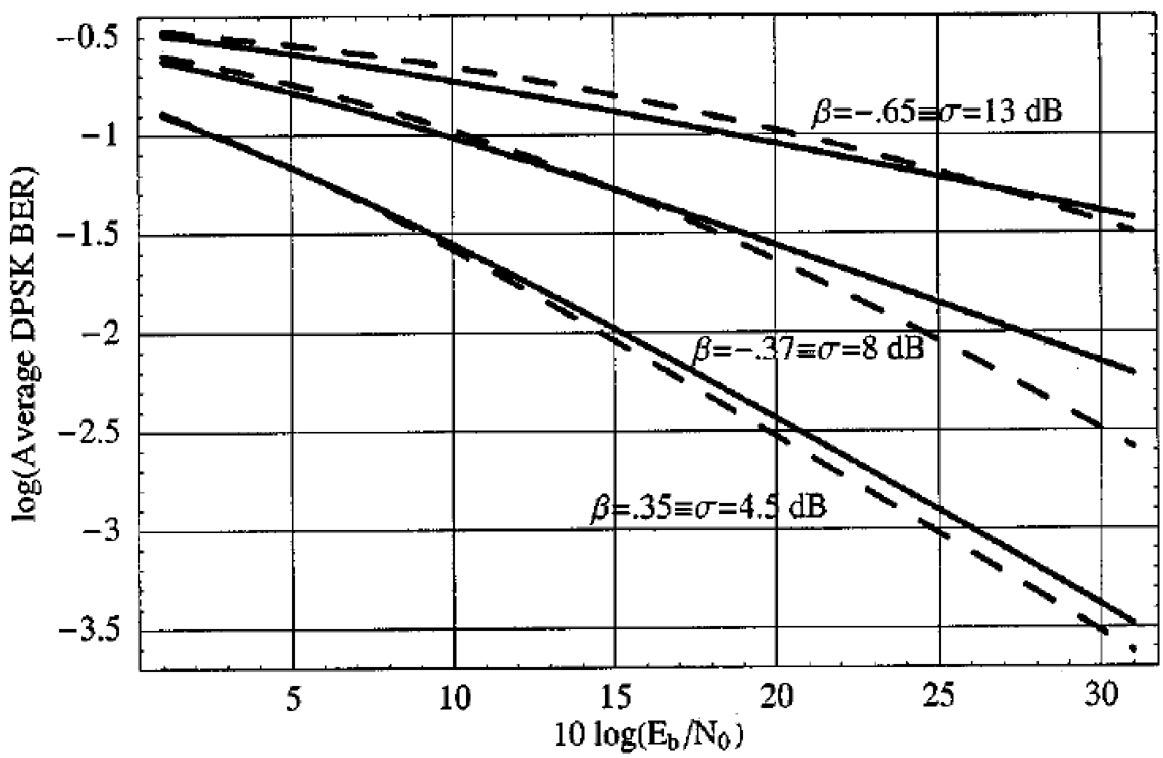

Fig. 1. Average bit error rates of DPSK for *K* and Rayleigh-lognormal distributions, assuming different values for the fading parameter $\beta(\equiv\sigma)$.

——— *K* fading

- - - - - Rayleigh-lognormal fading

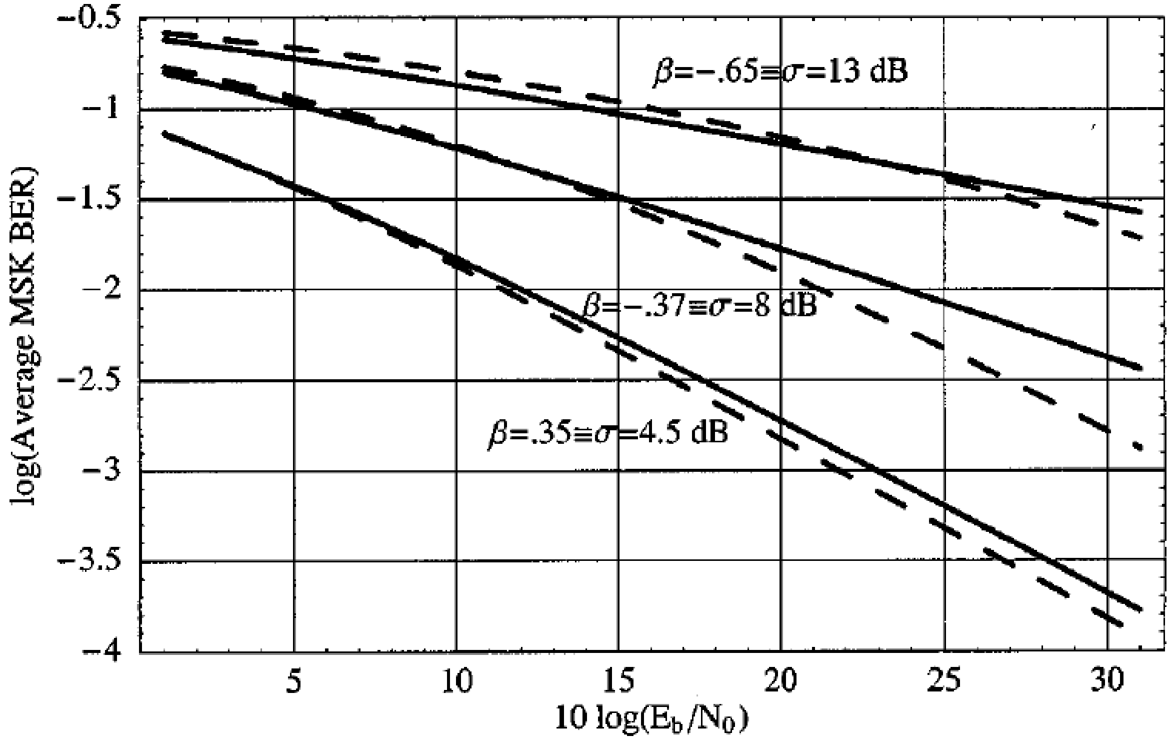

Fig. 2. Average bit error rates of MSK for *K* and Rayleigh-lognormal distributions, assuming different values for the fading parameter $\beta(\equiv \sigma)$.

——— *K* fading

- - - - - Rayleigh-lognormal fading