Expected Number of Maxima in the Envelope of a Spherically Invariant Random Process

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***Abstract*—In many engineering applications, specially in communication engineering, one usually encounters a bandpass non-Gaussian random process, with a slowly varying envelope. Among the available models for non-Gaussian random processes, spherically invariant random processes (SIRP's) play an important role. These processes are of interest mainly due to the fact that they allow one to relax the assumption of Gaussianity, while keeping many of its useful characteristics. In this paper, we have derived a simple and closed-form formula for the expected number of maxima of a SIRP envelope. Since Gaussian random processes are special cases of SIRP's, this formula holds for Gaussian random processes as well. In contrast with the available complicated expression for the expected number of maxima in the envelope of a Gaussian random process, our simple result holds for an arbitrary power spectrum. The key idea in deriving this result is the application of the characteristic function, rather than the probability density function, for calculating the expected level crossing rate of a random process.**

***Index Terms*—Characteristic function, Envelope, Gaussian processes, Level crossing problems, Maxima of the envelope, Spherically invariant processes.**

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I. INTRODUCTION

In many engineering applications, specially in communication engineering, we usually encounter a bandpass non-Gaussian random process. Such a random process behaves approximately like a single frequency sine wave, with slowly varying envelope and phase. Although both of the envelope and phase carry useful information about the original random process, however, the envelope usually receives more attention. So, it is important to study the statistical properties of the envelope of bandpass non-Gaussian random processes. In this paper we focus on the expected number of maxima of the envelope, per unit time. In addition to the communication engineering, this topic is also of interest of other fields, for example, mechanical and structural engineering [1] [2].

In order to calculate the expected number of maxima of the envelope, per unit time, we have to consider a model for the underlying bandpass non-Gaussian random process. Among the available models for non-Gaussian random processes, spherically invariant random processes (SIRP's) play an important role. These processes are of interest mainly due to the fact that they allow one to relax the assumption of Gaussianity, while keeping many of its useful characteristics [3]-[6]. Hence, it is not surprising why SIRP's have found numerous applications, summarized in [7]. In what follows, we assume that the underlying bandpass non-Gaussian random process is a bandpass SIRP, and focus on the expected number of the envelope maxima per unit time. As will be discussed later, Gaussian random processes are special cases of SIRP's. So, our results are applicable to Gaussian random processes as well.

According to the characterization theorem for SIRP's [3] [4], any SIRP can be represented by a Gaussian random process, multiplied by an independent and positive-valued random variable. Therefore, the wide sense stationary (WSS), bandpass, zero-mean SIRP $X(t)$ can be written as:

$$X(t) = C\,I(t), \tag{1}$$

where *C* is a positive random variable with the probability density function (PDF) $p_C(c)$, and $I(t)$ is a WSS, bandpass zero-mean Gaussian random process, independent of *C*. Using Rice's representation [8] [9] [10], $I(t)$ can be written as:

$$I(t) = I_c(t)\cos 2\pi f_m t - I_s(t)\sin 2\pi f_m t\,, \tag{2}$$

where f_m is a representative midband frequency, and $I_c(t)$ and $I_s(t)$ are two joint WSS, lowpass, and zero-mean Gaussian random processes. Writing (2) in polar form yields:

$$I(t) = R(t)\cos[2\pi f_m t + \Theta(t)], \tag{3}$$

where $R(t)$ and $\Theta(t)$ are the envelope and phase of $I(t)$, respectively, defined by:

$$R(t) = \sqrt{I_c^2(t) + I_s^2(t)} \quad , \quad \tan\Theta(t) = I_s(t)/I_c(t). \tag{4}$$

After replacing $I(t)$ in (1) with (3), $X(t)$ can be written as:

$$X(t) = C\,R(t)\cos[2\pi f_m t + \Theta(t)], \tag{5}$$

which shows that $C\,R(t)$ and $\Theta(t)$ are the envelope and phase of $X(t)$, respectively. Based on this representation for $X(t)$ in (5), it becomes evident that the envelopes of $X(t)$ and $I(t)$, i.e. $C\,R(t)$ and $R(t)$, respectively, have the same expected number of maxima per unit time. So, for calculating the expected number of maxima of the envelope of a bandpass SIRP, we only need to calculate the expected number of maxima of the corresponding Gaussian process envelope, per unit time. It should be mentioned that such a shortcut approach for deriving the statistical properties of SIRP's, based on their characterization in terms of Gaussian random processes, has already been used in [4] (see also [11]), to determine the expected number of zeros of SIRP's per unit time.

In the early days of statistical communication theory, Rice [8], Davenport and Root (see chap. 8 of [12]), and Middleton (see chap. 9 of [13]) derived basic statistical properties of $R(t)$. However, due to the complicated multivariate PDF and characteristic function (CF) of $R(t)$, only a limited number of its statistical properties have been explored and still there are numerous unsolved problems regarding $R(t)$. One of these unsolved problems is the expected number of maxima of $R(t)$ per unit time. In this paper we show that a CF-based approach, instead of the traditional PDF-based method, makes it possible to derive a simple and closed-form expression for the expected number of maxima of $R(t)$, per unit time, assuming an arbitrary power spectrum for $I(t)$.

The remainder of this paper is organized as follows. In Section II, the Rice's result for *N*, the expected number of maxima of the Gaussian process envelope, per unit time, is presented. His result has been derived for a power spectrum with even symmetry. Then it is shown in Section III that for an arbitrary power spectrum, PDF-based level crossing formula results in a double-fold integral for *N*, which should be solved numerically. Section IV presents a novel CF-based level crossing formula, by which an exact and simple solution is derived for N, assuming an arbitrary-shaped power spectrum. Section V gives the required result for $\tilde{N}$, the expected number of maxima of the SIRP envelope, per unit time, assuming an arbitrary power spectrum. Some concluding remarks are given in Section VI.

II. RICE'S RESULT FOR THE MAXIMA OF $R(t)$ WITH A SYMMETRIC SPECTRUM

Rice was the first person who studied the statistical properties of the maxima of $R(t)$ [8]. In order to obtain a closed-form formula for N, the expected number of maxima of $R(t)$ per unit time, he assumed that $w_I(f)$, the one-sided power spectrum of $I(t)$ defined for $f \geq 0$, has even symmetry around f_m, to simplify the problem. Then he derived the following formula using a PDF-based approach [8] (the method of derivation is briefly outlined in [14]):

$$N = \frac{(\gamma^2 - 1)^2}{(2\gamma)^{5/2}} \left(\frac{b_2}{\pi b_0} \right)^{1/2} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{n}{2} + \frac{5}{4})}{\Gamma(\frac{n}{2} + \frac{7}{4})} \frac{\delta_n}{\gamma^n}, \qquad (6)$$

where:

$$\delta_n = \sum_{k=0}^{n} \frac{(\frac{1}{2})(\frac{3}{2})...(k - \frac{1}{2})}{k!} (n - k + 1) b^k, \ \ \gamma^2 = \frac{b_0 b_4}{{b_2}^2}, \ \ b = \frac{3 - \gamma^2}{2}, \qquad (7)$$

and b_n is the nth central spectral moment of $I(t)$, given by:

$$b_n = (2\pi)^n \int_0^{\infty} (f - f_m)^n w_I(f) df \ . \qquad (8)$$

Note that (6) is valid only for $w_I(f)$ with even symmetry around f_m, which results in $b_1 = b_3 = 0$. Equation (6) has been used in pp. 162-163 of [15] for computing the mean distance between the envelope fadings in a Rayleigh fading channel, while in [16], it has been employed for estimating the bandwidth of a Gaussian random process with a Gaussian-shaped power spectrum. Some historical remarks about Rice's work on the maxima of $R(t)$ can be found in [17] and [18].

In the sequel, we discuss PDF and CF-based methods for calculating N, assuming an arbitrary shape for $w_I(f)$, which indicates that the associated b_n's are not necessarily zero for odd n. Note that such a general assumption is not just of theoretical interest, because in many practical cases, such as wireless propagation channels where the scattering is nonisotropic, the underlying random process has a nonsymmetric power spectrum [19].

III. PDF-Based Methods for N Assuming an Arbitrary Spectrum

According to the celebrated Rice's formula for the expected number times per unit time that a stationary random process $Y(t)$ crosses the level ℓ [14], we have:

$$E[N_{\ell}\{Y(t)\}]=\int_{-\infty}^{\infty}|y'|p_{YY'}(\ell,y')dy', \tag{9}$$

where prime denotes differentiation with respect to time t, and $p_{YY'}(y,y')$ is the joint PDF of the random variables $Y=Y(t_0)$ and $Y'=Y'(t_0)$, with t_0 as an arbitrary instant of time.

It is clear that every maximum of $R(t)$ corresponds to a zero in $R'(t)$ with negative slope. So, N is equal to the expected number of zeros of $R'(t)$ with negative slope, per unit time. Therefore, according to (9), we can write the following equation for N:

$$N=-\int_{-\infty}^{0}r''p_{R'R''}(0,r'')dr'', \tag{10}$$

where the random variables R' and R'' are defined by $R'(t_0)$ and $R''(t_0)$,[1] respectively. It has been shown in [20] that for an arbitrary power spectrum, R' and R'' are independent, simplifying (10) to:

$$N=-p_{R'}(0)\int_{-\infty}^{0}r''p_{R''}(r'')dr''. \tag{11}$$

So, we need to derive an expression for $p_{R''}(r'')$. For an arbitrary power spectrum, $p_{R''}(r'')$ can be expressed in terms of a single-fold integral [20]. Hence, as a general result, we conclude that a double-fold integral has to be calculated numerically, in order to determine the numerical value of N in (11). For even-symmetric power spectra, other double-fold integrals are derived in [21] and [22] for N. Of course another alternative for even-symmetric power spectra is eq. (6), as discussed before.

IV. CF-Based Method for N Assuming an Arbitrary Spectrum

The celebrated Rice's formula for the expected level crossing rate of a stationary random process $Y(t)$ has originally been expressed in terms of $p_{YY'}(y,y')$, and it is traditional to compute $E[N_{\ell}\{Y(t)\}]$ according to (9). However, there are many cases where $\Phi_{YY'}(\omega_1,\omega_2)=E[e^{j\omega_1 Y+j\omega_2 Y'}]$, the joint CF of the random variables Y and Y' with $j^2=-1$, can be derived more easily than $p_{YY'}(y,y')$. For example,

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[1] It is shown in [20] that the random process $R''(t)$ does not exist in the mean square sense. However, it exists almost everywhere (with probability one). Anyway, in the sequel we will define and work with a process which has the same expected number of maxima of $R(t)$, but is also differentiable in the mean square sense, up to the order that we need. Therefore, still we will be able to use the more convenient concepts and tools of the mean square stochastic calculus.

in calculating the fading rate in diversity systems operating over frequency selective fading channels, it is very straightforward to derive $\Phi_{YY'}(\omega_1,\omega_2)$ rather than $p_{YY'}(y,y')$ [26]. So, we need to express (9) in terms of $\Phi_{YY'}(\omega_1,\omega_2)$. Such a formula apparently was first reported in [27] and then rederived, independently, in [26]:

$$E[N_\ell\{Y(t)\}] = \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega_2}\frac{d\,\Phi_{YY'}(\omega_1,\omega_2)}{d\omega_2}\; e^{-j\omega_1\ell}d\omega_1 d\omega_2 = \frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Phi_{YY'}(\omega_1,\omega_2)-\Phi_Y(\omega_1)}{\omega_2{}^2}e^{-j\omega_1\ell}d\omega_1 d\omega_2. \tag{12}$$

The same results can be obtained using Hsu's formula for the absolute moments of a random variable, in terms of the associated CF, given in pp. 19-20 of [28]. This is due to the fact that (9) is essentially a first-order absolute moment, i.e., $E[N_\ell\{Y(t)\}] = p_Y(\ell)\int_{-\infty}^{\infty}|y'|\,p_{Y'|Y}(y'\,|\,\ell)dy'$. Note that because of the presence of ω_2 in the denominator of the integrands in (12), both integrals seem to be divergent in some cases. However, in Appendix A it has been shown that both integrands are bounded at the origin $(\omega_1,\omega_2)=(0,0)$, provided that $E[Y'^2]$ is finite.

By symmetry, *N* is equal to the half of the expected number of the extrema of $R(t)$ per unit time. In other words, *N* is equal to the half of the expected number of zeros of $R'(t)$ per unit time. Therefore we rewrite (10) as:

$$N = \frac{1}{2}\int_{-\infty}^{\infty}|r''|\,p_{R'R''}(0,r'')dr'' = \frac{1}{2}E[N_0\{R'(t)\}].$$

Since $R(t)$ has a Rayleigh PDF, i.e. $p_R(r) = r\exp\left(-r^2/2b_0\right)/b_0$ for $r>0$ and zero otherwise [8], we have $\Pr\{R(t)\le 0\}=0$. So, we can conclude that $R'(t)$ and the auxiliary random process $A(t)=R(t)R'(t)$ have the same number of zeros per unit time, i.e., $N_0\{R'(t)\}=N_0\{A(t)\}$. In this way, the following useful relationship can be established:

$$N = (1/2)E[N_0\{A(t)\}]. \tag{13}$$

The utility of the auxiliary random process $A(t)$, also used in [29], lies on the fact that $\Phi_{AA'}(\omega_1,\omega_2)$, in which $A=A(t_0)$ and $A'=A'(t_0)$,[2] with t_0 as an arbitrary instant of time, can be calculated more easily than $\Phi_{R'R''}(\omega_1,\omega_2)$.

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[2] Since $A'(t)=R'^2(t)+R(t)R''(t)$, we see that the random process $A'(t)$ exists almost everywhere (with probability one), as $R'(t)$ and $R''(t)$ exist in the same sense [20]. However, note that $A(t)=\frac{1}{2}dR^2(t)/dt$ and $R^2(t)$ is shown to be twice differentiable in the mean square sense [20]. Therefore, $A'(t)$ exists in the mean square sense.

In Appendix B, the following simple closed-form expression has been derived for $\Phi_{AA'}(\omega_1,\omega_2)$:

$$\Phi_{AA'}(\omega_1,\omega_2)=\frac{1}{1+\alpha_1\omega_1^{\ 2}+\alpha_2\omega_2^{\ 2}-j2B\omega_2^{\ 3}}, \tag{14}$$

where:

$$\begin{aligned}\alpha_1&=b_0b_2-b_1^{\ 2},\\ \alpha_2&=b_0b_4+3b_2^{\ 2}-4b_1b_3,\\ B&=b_0b_2b_4+2b_1b_2b_3-b_2^{\ 3}-b_0b_3^{\ 2}-b_1^{\ 2}b_4.\end{aligned} \tag{15}$$

According to the first equation in (12):

$$E[N_0\{A(t)\}]=\frac{-1}{2\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\omega_2}\frac{d\Phi_{AA'}(\omega_1,\omega_2)}{d\omega_2}d\omega_1 d\omega_2\,. \tag{16}$$

Integration with respect to ω_1 yields (see p. 292 of [30]):

$$\int_{-\infty}^{\infty}\Phi_{AA'}(\omega_1,\omega_2)d\omega_1=\frac{\pi}{\sqrt{\alpha_1}}\frac{1}{\sqrt{1+\alpha_2\omega_2^{\ 2}-j2B\omega_2^{\ 3}}}\,.$$

After differentiation with respect to ω_2, (16) reduces to:

$$E[N_0\{A(t)\}]=\frac{1}{2\pi\sqrt{\alpha_1}}\int_{-\infty}^{\infty}\frac{\alpha_2-j3B\omega_2}{\left(1+\alpha_2\omega_2^{\ 2}-j2B\omega_2^{\ 3}\right)^{3/2}}d\omega_2\,. \tag{17}$$

By introducing the new parameter β:

$$\beta=\frac{-B}{\alpha_2^{\ 3/2}}=\frac{-b_0b_2b_4-2b_1b_2b_3+b_2^{\ 3}+b_0b_3^{\ 2}+b_1^{\ 2}b_4}{\left(b_0b_4+3b_2^{\ 2}-4b_1b_3\right)^{3/2}}, \tag{18}$$

and also using the new variable $\zeta=\sqrt{\alpha_2}\,\omega_2$, (17) can be written as:

$$E[N_0\{A(t)\}]=\frac{\sqrt{\alpha_2}}{2\pi\sqrt{\alpha_1}}\int_{-\infty}^{\infty}\frac{1+j3\beta\zeta}{\left(1+\zeta^2+j2\beta\zeta^3\right)^{3/2}}d\zeta\,. \tag{19}$$

After writing $1+j3\beta\zeta$ and $1+\zeta^2+j2\beta\zeta^3$ in polar forms, it can be easily shown that:

$$E[N_0\{A(t)\}]=\frac{K(\beta)}{\pi}\sqrt{\frac{\alpha_2}{\alpha_1}}, \tag{20}$$

where:

$$K(\beta)=\int_0^{\infty}\frac{\left(1+9\beta^2\zeta^2\right)^{1/2}}{\left(1+2\zeta^2+\zeta^4+4\beta^2\zeta^6\right)^{3/4}}\cos\left[\tan^{-1}\left(3\beta\zeta\right)-\frac{3}{2}\tan^{-1}\left(\frac{2\beta\zeta^3}{1+\zeta^2}\right)\right]d\zeta\,. \tag{21}$$

Finally, substitution of (20) into (13) gives our main result:

$$N=\frac{K(\beta)}{2\pi}\sqrt{\frac{\alpha_2}{\alpha_1}}\,. \tag{22}$$

Comparison of (22) with (6), and also the double-fold integrals given in [21] and [22], reveals that (22), which is not restricted to even-symmetric power spectra, is more convenient for both numerical calculations and analytic manipulations.

It has been shown in Appendix C that for all possible $w_I(f)$'s, the parameter β varies over a finite range:

$$-\sqrt{3}/9\le\beta\le 1/8\,. \tag{23}$$

For $\beta=0$ we have $K(0)=1$. The plot of $K(\beta)$, an even function with respect to $\beta=0$, over the above range of β exhibits that:

$$1.00\le K(\beta)\le 1.05. \tag{24}$$

Hence, instead of the complicated integral in (21), $K(\beta)$ may be accurately approximated by a polynomial. For example, the following polynomial is a good approximation with an absolute error less than or equal to 0.0001:

$$K(\beta)\approx 0.9999+0.0463|\beta|+2.544\beta^2-12.19|\beta|^3+22.7\beta^4. \tag{25}$$

Interestingly, for even-symmetric power spectra, $K(\beta)$ of this paper is equal to the parameter k [21], defined in the caption of Fig. 6. In [21], which has focused only on even-symmetric power spectra, k is plotted versus $x=(b_4b_0-b_2^2)/b_2^2$ in Fig. 6. Using (18), it is easy to verify that $\beta=-x/(x+4)^{3/2}$ for even-symmetric power spectra.

Based on the definitions of α_1, α_2, and β in (15) and (18), we see that they are functions of b_n's, and b_n's depend upon f_m via (8). This may mislead us to consider N in (22) as a function of f_m. However, as has been discussed in [31], $R(t)$ in (4) is independent of the midband frequency f_m. So, we expect N to be independent of f_m. To verify this fact, we need to consider the nth noncentral spectral moment of $I(t)$:

$$\mu_n = (2\pi)^n \int_0^\infty f^n w_I(f) df \,. \tag{26}$$

Note that μ_n's are independent of f_m. By expanding the term $(f - f_m)^n$ in (8) according to the binomial expansion in p. 21 of [30], we obtain:

$$b_n = n! \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (-2\pi f_m)^k \mu_{n-k} \,. \tag{27}$$

After replacing b_n's in α_1, α_2, and β with μ_n's according to (27), we obtain:

$$\begin{aligned} \alpha_1 &= \mu_0 \mu_2 - \mu_1^{\,2}, \\ \alpha_2 &= \mu_0 \mu_4 + 3\mu_2^{\,2} - 4\mu_1 \mu_3, \\ \beta &= \frac{-\mu_0 \mu_2 \mu_4 - 2\mu_1 \mu_2 \mu_3 + \mu_2^{\,3} + \mu_0 \mu_3^{\,2} + \mu_1^{\,2} \mu_4}{\left(\mu_0 \mu_4 + 3\mu_2^{\,2} - 4\mu_1 \mu_3\right)^{3/2}} \,. \end{aligned} \tag{28}$$

Hence, α_1, α_2, and β are free of f_m, making N in (22) independent of f_m. It may be worth reminding that although f_m has no effect on N, yet it may affect other properties of the envelope. For example, we obtain the optimum complex envelope (see pp. 367-368 of [10]), if we choose f_m to be the center of gravity of $w_I(f)$, i.e. $f_m = \mu_1 / \mu_0$. Such a selection leads to $b_1 = 0$, which in turn reduces the number of terms in α_1, α_2, β, and consequently in N.

Some numerical examples are given in Table I, where several $w_I(f)$'s with different mathematical forms have been listed, and the associated α_1, α_2, β, $K(\beta)$, and N have been computed according to (28), (21), and (22). Note that for numerical calculations, (28) is preferable to (15) and (18), because μ_n's are independent of f_m, and we do not need to specify a value for f_m. As the noncentral moments of a PDF can be expressed in terms of the Mellin transform of the PDF, discussed in p. 74 of [32], μ_n's can be written as the Mellin transform of $w_I(f)$ as well:

$$\mu_n = (2\pi)^n M\{w_I(f), n+1\}, \tag{29}$$

where $M\{q(\xi), s\}$ is the Mellin transform of $q(\xi)$, i.e. $M\{q(\xi), s\} = \int_0^\infty \xi^{s-1} q(\xi) d\xi$. As a result, for many choices of $w_I(f)$, closed-form expressions for μ_n's can be readily obtained from the extensive available tables of the Mellin transform [33] [34].

For comparison purposes, it is instructive to note that the $w_I(f)$'s in the first and the fourth rows of Table I have even symmetry around $f_m = 0.5$ and $f_m = 1$, respectively. So, if we compute N via the Rice's result in (6), for the above f_m's, with b_n's calculated according to (27) based on the μ_n's given in Table I, we obtain the same results for N listed in Table I.

V. EXPECTED NUMBER OF MAXIMA OF THE SIRP ENVELOPE

As mentioned in Section I, the envelopes of the SIRP $X(t)$ and its associated Gaussian random process $I(t)$ in (1), i.e. $C R(t)$ and $R(t)$, respectively, have the same expected number of maxima per unit time. Hence, $\tilde{N} = N$, where $\tilde{N}$ is the expected number of maxima for the envelope of $X(t)$ per unit time and N is given by (22). In order to express $\tilde{N}$ in terms of $\tilde{b}_n$'s, the central spectral moments of $X(t)$, rather than b_n's, we proceed as follows. By definition:

$$\tilde{b}_n = (2\pi)^n \int_0^\infty (f - f_m)^n w_X(f) df . \tag{30}$$

Since $X(t)$ is related to $I(t)$ through (1), we have $w_X(f) = E[C^2] w_I(f)$, and then $\tilde{b}_n = E[C^2] b_n$. Now, rewriting (22) in terms of $\tilde{b}_n$'s yields the final result:

$$\tilde{N} = \frac{K(\tilde{\beta})}{2\pi} \sqrt{\frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}} , \tag{31}$$

where $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\beta}$ are the same as α_1, α_2, and β in (15) and (18), with b_n's replaced by $\tilde{b}_n$'s.

VI. CONCLUSION

Spherically invariant random processes (which include Gaussian random processes as special cases) play an important role in modeling non-Gaussian random processes. They allow us to relax the assumption of Gaussianity, while keeping many of its useful characteristics. In this paper we have derived a simple, exact, and closed-form expression for the expected number of maxima of a SIRP envelope, per unit time, assuming an arbitrary-shaped power spectrum for the underlying SIRP. We have obtained this result using the characteristic function-based level crossing formula, while the traditional probability density function-based level crossing formula leads to a complicated double-fold integral, which has to be solved via numerical integration techniques. As an application, the interested reader may refer to [35], where the formula for the expected number of maxima of the SIRP envelope, derived in this paper, has been used for estimating the velocity of a mobile vehicle in a Rayleigh fading channel.

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TABLE I

NUMERICAL VALUES OF N FOR SEVERAL POWER SPECTRA

	$w_I(f)$	$\mu_n/(2\pi)^n$	α_1	α_2	β	$K(\beta)$	N
1	$\begin{cases}1, 0<f<1\\0, f>1\end{cases}$	$(n+1)^{-1}$	3.290	51.95	-76.07 $E-3$	1.014	0.6410
2	$\begin{cases}f, 0<f<1\\0, f>1\end{cases}$	$(n+2)^{-1}$	0.5483	6.494	-86.07 $E-3$	1.016	0.5566
3	$\begin{cases}f^4, 0<f<1\\0, f>1\end{cases}$	$(n+5)^{-1}$	0.03133	0.1767	-93.17 $E-3$	1.018	0.3848
4	$\begin{cases}f, 0<f<1\\2-f, 1<f<2\\0, f>2\end{cases}$	$\dfrac{2(2^{n+1}-1)}{(n+1)(n+2)}$	6.580	233.8	-111.6 $E-3$	1.023	0.9707
5	$\exp(-f)$	$n!$	39.48	18702.5	-96.23 $E-3$	1.019	3.530

APPENDIX A

THE BEHAVIOR OF THE INTEGRANDS IN (12) AT THE ORIGIN

According to [36], p. 45, the Taylor expansion of the joint CF of the random variables Y and Y' in the neighborhood of the origin is given by:

$$\Phi_{YY'}(\omega_1,\omega_2)=1+\sum_{\nu_1+\nu_2\le 2}\eta_{\nu_1,\nu_2}\frac{j^{\nu_1+\nu_2}\omega_1^{\nu_1}\omega_2^{\nu_2}}{\nu_1!\nu_2!}+O((\omega_1^2+\omega_2^2)^{3/2}),\ \text{as}\ (\omega_1,\omega_2)\to(0,0),$$

where $\eta_{\nu_1,\nu_2}=E[Y^{\nu_1}Y'^{\nu_2}]$ and for two functions f_1 and f_2, $f_1=O(f_2)$ over a neighborhood means that there exits a constant $F_0>0$ such that $|f_1|\le F_0|f_2|$ over that neighborhood. By expanding the summation we get:

$$\Phi_{YY'}(\omega_1,\omega_2)=1+\eta_{1,0}j\omega_1+\eta_{0,1}j\omega_2-\eta_{1,1}\omega_1\omega_2-\frac{\eta_{2,0}}{2}\omega_1^2-\frac{\eta_{0,2}}{2}\omega_2^2+O((\omega_1^2+\omega_2^2)^{3/2}),\ \text{as}\ (\omega_1,\omega_2)\to(0,0).$$

Let $Y(t)$ be a stationary random process which is differentiable in the mean square sense. This implies that $\eta_{0,1}=E[Y']=0$. Furthermore, $\eta_{1,1}=E[YY']=E[YE[Y'|Y]]=0$ as $E[Y'|Y]=0$ [37] [38]. The above expansion now can be written as:

$$\Phi_{YY'}(\omega_1,\omega_2)=1+jE[Y]\omega_1-\frac{E[Y^2]}{2}\omega_1^2-\frac{E[Y'^2]}{2}\omega_2^2+O((\omega_1^2+\omega_2^2)^{3/2}),\ \text{as}\ (\omega_1,\omega_2)\to(0,0),$$

in which $E[Y]$, $E[Y^2]$, and $E[Y'^2]$ are finite. This expansion yields the following results for the integrands in (12):

$$\frac{1}{\omega_2}\frac{d\,\Phi_{YY'}(\omega_1,\omega_2)}{d\omega_2}e^{-j\omega_1\ell}\Bigg|_{(\omega_1,\omega_2)=(0,0)}=-E[Y'^2],$$

$$\frac{\Phi_{YY'}(\omega_1,\omega_2)-\Phi_Y(\omega_1)}{\omega_2{}^2}e^{-j\omega_1\ell}\Bigg|_{(\omega_1,\omega_2)=(0,0)}=-\frac{E[Y'^2]}{2}.$$

Obviously, as long as $E[Y'^2]$ is finite, both the integrands in (12) are bounded at the origin.

APPENDIX B

DERIVATION OF THE JOINT CHARACTERISTIC FUNCTION OF A AND A'

A. Direct Method

Since $I_c(t)$ and $I_s(t)$ in (2) are two joint Gaussian random processes, their time derivatives $I_c'(t)$, $I_s'(t)$, $I_c''(t)$, and $I_s''(t)$ are also joint Gaussian random processes. For an arbitrary but fixed instant of time t_0, the six dimensional random vector $\mathbf{V} = [I_c \;\; I_s' \;\; I_c'' \;\; I_s \;\; I_c' \;\; I_s'']^T$, with $I_c = I_c(t_0)$, $I_s' = I_s'(t_0)$, $I_c'' = I_c''(t_0)$, $I_s = I_s(t_0)$, $I_c' = I_c'(t_0)$, $I_s'' = I_s''(t_0)$ and T as the transpose operator, is a Gaussian random vector with zero mean-vector and the following covariance-matrix [8]:

$$\mathbf{M} = \begin{bmatrix} b_0 & b_1 & -b_2 & 0 & 0 & 0 \\ b_1 & b_2 & -b_3 & 0 & 0 & 0 \\ -b_2 & -b_3 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_0 & -b_1 & -b_2 \\ 0 & 0 & 0 & -b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -b_2 & b_3 & b_4 \end{bmatrix}, \tag{A.1a}$$

where b_n's are defined in (8).

Based on the definition of $R(t)$ in (4), it can be shown that for $A(t) = R(t)R'(t)$ we have:

$$A(t) = I_c(t)I_c'(t) + I_s(t)I_s'(t),$$

$$A'(t) = I_c(t)I_c''(t) + I_c'^2(t) + I_s(t)I_s''(t) + I_s'^2(t).$$

So, for $t = t_0$ we have:

$$\Phi_{AA'}(\omega_1, \omega_2) = E[e^{j\omega_1(I_cI_c' + I_sI_s') + j\omega_2\left(I_cI_c'' + I_c'^2 + I_sI_s'' + I_s'^2\right)}],$$

where $A = A(t_0)$ and $A' = A'(t_0)$. Note that the expression in the exponent is a quadratic form. By constructing the symmetric matrix $\mathbf{D}$ as discussed in p. 264 of [39]:

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \omega_2/2 & 0 & \omega_1/2 & 0 \\ 0 & \omega_2 & 0 & \omega_1/2 & 0 & 0 \\ \omega_2/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_1/2 & 0 & 0 & 0 & \omega_2/2 \\ \omega_1/2 & 0 & 0 & 0 & \omega_2 & 0 \\ 0 & 0 & 0 & \omega_2/2 & 0 & 0 \end{bmatrix}, \tag{A.1b}$$

this quadratic form can be written as $\mathbf{V}^T\mathbf{D}\mathbf{V}$. So:

$$\Phi_{AA'}(\omega_1,\omega_2) = E[e^{j\mathbf{V}^T\mathbf{D}\mathbf{V}}] = \Phi_{\mathbf{V}^T\mathbf{D}\mathbf{V}}(1)\,. \tag{A.2}$$

It has been shown in p. 152 of [40] that for a $k \times 1$ real Gaussian random vector $\mathbf{Z}$ with zero mean-vector and covariance-matrix $\mathbf{\Sigma}$, the CF of the homogeneous quadratic form $\mathbf{Z}^T\mathbf{F}\mathbf{Z}$ can be expressed as:

$$\Phi_{\mathbf{Z}^T\mathbf{F}\mathbf{Z}}(\omega) = \prod_{i=1}^{k} \frac{1}{\sqrt{1 - j2\omega\lambda_i}}, \tag{A.3}$$

where $\mathbf{F}$ is the $k \times k$ real symmetric matrix of the quadratic form, and λ_i's are the eigenvalues of $\mathbf{\Sigma F}$, i.e. the roots of equation $\det(\mathbf{\Sigma F} - \lambda \mathbf{I}) = 0$, with $\mathbf{I}$ as the $k \times k$ identity matrix. However, as is stated in [41], (A.3) can be written as:

$$\Phi_{\mathbf{Z}^T\mathbf{F}\mathbf{Z}}(\omega) = \frac{1}{\sqrt{\det(\mathbf{I} - j2\omega\mathbf{\Sigma F})}}, \tag{A.4}$$

which is more appropriate for our purpose. For the case of a complex Gaussian random vector $\mathbf{Z}$ and a complex Hermitian $\mathbf{F}$, the reader may refer to pp. 590-595 of [42].

Based on the above information, $\Phi_{AA'}(\omega_1,\omega_2)$ in (A.2) can be written as:

$$\Phi_{AA'}(\omega_1,\omega_2) = \frac{1}{\sqrt{\det(\mathbf{I} - j2\mathbf{MD})}}\,.$$

Using Mathematica software [43] for symbolic operations, it can be shown that:

$$\det(\mathbf{I} - j2\mathbf{MD}) = \left(1 + \alpha_1\omega_1{}^2 + \alpha_2\omega_2{}^2 - j2B\omega_2{}^3\right)^2,$$

which in turn results in (14).

B. Indirect Method

Based on the definition of the derivative for a random process $Y(t)$ we have:

$$\Phi_{YY'}(\omega_1,\omega_2) = E\left[\lim_{\tau\to 0} \exp\left(j\omega_1 Y + j\omega_2 \frac{Y_\tau - Y}{\tau}\right)\right],$$

where $Y_\tau = Y(t_0 + \tau)$. Based on the bounded convergence theorem in p. 108 of [44], we can change the order of lim and E to obtain:

$$\Phi_{YY'}(\omega_1,\omega_2) = \lim_{\tau\to 0} \Phi_{YY_\tau}\left(\omega_1 - \frac{\omega_2}{\tau}, \frac{\omega_2}{\tau}\right). \tag{A.5}$$

So, $\Phi_{AA'}(\omega_1,\omega_2)$ can be derived by calculating the limit of $\Phi_{AA_\tau}(\omega_1-\omega_2/\tau,\omega_2/\tau)$, with $A_\tau = A(t_0+\tau)$.

The joint CF of A and A_τ can be written as:

$$\Phi_{AA_\tau}(\omega_1,\omega_2) = E[e^{j\omega_1(I_c I'_c + I_s I'_s)+j\omega_2(I_{c\tau}I'_{c\tau}+I_{s\tau}I'_{s\tau})}],$$

in which $I_{c\tau} = I_c(t_0+\tau)$, $I'_{c\tau} = I'_c(t_0+\tau)$, $I_{s\tau} = I_s(t_0+\tau)$, and $I'_{s\tau} = I'_s(t_0+\tau)$. Obviously, the expression in the exponent is a quadratic form. By constructing the symmetric matrix $\mathbf{C}$ as follows [39]:

$$\mathbf{C} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_2 \\ \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.6}$$

and also defining $\mathbf{U} = [I_c \ \ I_{c\tau} \ \ I_s \ \ I_{s\tau} \ \ I'_c \ \ I'_{c\tau} \ \ I'_s \ \ I'_{s\tau}]^T$, this quadratic form can be written as $\mathbf{U}^T\mathbf{C}\mathbf{U}$. Therefore we have:

$$\Phi_{AA_\tau}(\omega_1,\omega_2) = E[e^{j\mathbf{U}^T\mathbf{C}\mathbf{U}}] = \Phi_{\mathbf{U}^T\mathbf{C}\mathbf{U}}(1). \tag{A.7}$$

The random vector $\mathbf{U}$ is an eight dimensional Gaussian random vector with zero mean-vector and the following covariance-matrix [45]:

$$\mathbf{L} = \begin{bmatrix} b_0 & g & 0 & h & 0 & \dot{g} & b_1 & \dot{h} \\ g & b_0 & -h & 0 & -\dot{g} & 0 & \dot{h} & b_1 \\ 0 & -h & b_0 & g & -b_1 & -\dot{h} & 0 & \dot{g} \\ h & 0 & g & b_0 & -\dot{h} & -b_1 & -\dot{g} & 0 \\ 0 & -\dot{g} & -b_1 & -\dot{h} & b_2 & -\ddot{g} & 0 & -\ddot{h} \\ \dot{g} & 0 & -\dot{h} & -b_1 & -\ddot{g} & b_2 & \ddot{h} & 0 \\ b_1 & \dot{h} & 0 & -\dot{g} & 0 & \ddot{h} & b_2 & -\ddot{g} \\ \dot{h} & b_1 & \dot{g} & 0 & -\ddot{h} & 0 & -\ddot{g} & b_2 \end{bmatrix}, \tag{A.8}$$

where dot denotes differentiation with respect to τ, $g(\tau) = E[I_c(t)I_c(t+\tau)] = E[I_s(t)I_s(t+\tau)]$ is the autocorrelation function of both $I_c(t)$ and $I_s(t)$ in (2), and $h(\tau) = E[I_c(t)I_s(t+\tau)] = -E[I_s(t)I_c(t+\tau)]$

is the crosscorrelation function between $I_c(t)$ and $I_s(t)$. According to [8], $g(\tau)$ and $h(\tau)$ can easily be expressed in terms of $w_I(f)$:

$$g(\tau) = \int_0^\infty w_I(f)\cos[2\pi(f - f_m)\tau]df \ ,$$

$$h(\tau) = \int_0^\infty w_I(f)\sin[2\pi(f - f_m)\tau]df \ .$$

Based on (A.4), $\Phi_{AA_\tau}(\omega_1, \omega_2)$ in (A.7) can be written as $1\big/\sqrt{\det(\mathbf{I} - j2\mathbf{LC})}$. After calculating the limit of this expression according to (A.5) using Mathematica, and based on the following relations:

$$g(0) = b_0,\ \dot{g}(0) = 0,\ \ddot{g}(0) = -b_2,\ \dddot{g}(0) = 0,\ \ g^{(4)}(0) = b_4,$$
$$h(0) = 0,\ \dot{h}(0) = b_1,\ \ddot{h}(0) = 0,\ \dddot{h}(0) = -b_3,\ h^{(4)}(0) = 0,$$

the accuracy of (14) can be verified.

C. Comparison with a Simple Known Result

It is stated in pp. 195-196 of [46] that the product of two independent random variables with Rayleigh and Gaussian PDF's has the Laplace PDF. Since R and R' have these properties as well [16] [23]-[25], their product, A, has to be a Laplace random variable. For $\omega_1 = \omega$ and $\omega_2 = 0$ in (14), we obtain the CF of A:

$$\Phi_A(\omega) = \Phi_{AA'}(\omega, 0) = \frac{1}{1 + \alpha_1\omega^2},$$

which is the CF of a Laplace random variable (see p. 23 of [40]).

Appendix C

The Range of the Parameter β

As shown in (28), the parameter β does not depend upon the midband frequency f_m. For $f_m = \mu_1/\mu_0$, with μ_n's defined in (26), we obtain $b_1 = 0$ and β in (18) simplifies to:

$$\beta = \frac{b_2{}^3 + b_0 b_3{}^2 - b_0 b_2 b_4}{\left(b_0 b_4 + 3b_2{}^2\right)^{3/2}} . \tag{A.9}$$

Based on the Schwarz inequality we have:

$$\left[\int_0^\infty (f - f_m)^3 w_I(f)df\right]^2 \le \int_0^\infty (f - f_m)^2 w_I(f)df \int_0^\infty (f - f_m)^4 w_I(f)df ,$$

which can be written in terms of b_n's as $b_3^2 \le b_2 b_4$. Since b_0 is non-negative, we obtain:

$0 \le b_0 b_3^2 \le b_0 b_2 b_4$.

Application of this inequality to (A.9) yields:

$$\frac{b_2{}^3 - b_0 b_2 b_4}{\left(b_0 b_4 + 3b_2{}^2\right)^{3/2}} \le \beta \le \frac{b_2{}^3}{\left(b_0 b_4 + 3b_2{}^2\right)^{3/2}} ,$$

or:

$$\frac{1-\gamma^2}{\left(3+\gamma^2\right)^{3/2}} \le \beta \le \frac{1}{\left(3+\gamma^2\right)^{3/2}} , \tag{A.10}$$

with γ defined in (7). Based on the Schwarz inequality in the following form:

$$\left[\int_0^\infty (f - f_m)^2 w_I(f)df\right]^2 \le \int_0^\infty w_I(f)df \int_0^\infty (f - f_m)^4 w_I(f)df ,$$

it can be concluded that $\gamma^2 \ge 1$. The maximum of the right-hand side of (A.10) is equal to $1/8$, which occurs at $\gamma^2 = 1$; while for $\gamma^2 = 9$ the left-hand side of (A.10) takes its minimum value, which is equal to $-\sqrt{3}/9$. Therefore, for an arbitrary-shaped power spectrum we have:

$$\frac{-\sqrt{3}}{9} \le \beta \le \frac{1}{8}.$$

REFERENCES

[1] Y. K. Lin, "Random processes," *Appl. Mechanics Rev.*, vol. 22, pp. 825-831, 1969.

[2] D. E. Newland, *An Introduction to Random Vibrations and Spectral Analysis*, 2nd ed., New York: Longman, 1984.

[3] K. Yao, "A representation theorem and its applications to spherically-invariant random processes," *IEEE Trans. Inform. Theory*, vol. 19, pp. 600-608, 1973.

[4] G. L. Wise and N. C. Gallagher, Jr., "On spherically invariant random processes," *IEEE Trans. Inform. Theory*, vol. 24, pp. 118-120, 1978.

[5] K. Chu, "Estimation and decision for linear systems with elliptical random processes," *IEEE Trans. Automat. Contr.*, vol. 18, pp. 499-505, 1973.

[6] B. Picinbono, *Random Signals and Systems*. Englewood Cliffs, NJ: Prentice Hall, 1993.

[7] A. Abdi, H. A. Barger, and M. Kaveh, "Signal modeling in wireless fading channels using spherically invariant processes," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Istanbul, Turkey, 2000, pp. 2997-3000.

[8] S. O. Rice, "Mathematical analysis of random noise," reprinted in *Selected Papers on Noise and Stochastic Processes*. N. Wax, Ed., New York: Dover, 1954, pp. 133-294.

[9] W. A. Gardner, *Introduction to Random Processes with Applications to Signals and Systems*, 2nd ed., Singapore: McGraw-Hill, 1990.

[10] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed., Singapore: McGraw-Hill, 1991.

[11] G. L. Wise, "A comment on 'Zero-crossing rates of functions of Gaussian processes,'" *IEEE Trans. Inform. Theory*, vol. 38, p. 213, 1992.

[12] W. B. Davenport, Jr. and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*. New York: McGraw-Hill, 1958.

[13] D. Middleton, *An Introduction to Statistical Communication Theory*. New York: McGraw-Hill, 1960.

[14] I. F. Blake and W. C. Lindsey, "Level-crossing problems for random processes," *IEEE Trans. Inform. Theory*, vol. 19, pp. 295-315, 1973.

[15] M. D. Yacoub, *Foundations of Mobile Radio Engineering*. Boca Raton, FL: CRC, 1993.

[16] J. Komaili, L. A. Ferrari, and P. V. Sankar, "Estimating the bandwidth of a normal process from the level crossings of its envelope," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 1481-1483, 1987.

[17] D. Middleton, "S. O. Rice and the theory of random noise: Some personal recollections," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1367-1373, 1988.

[18] A. J. Rainal, "Origin of Rice's formula," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1383-1387, 1988.

[19] A. Abdi, J. A. Barger, and M. Kaveh, "A parametric model for the distribution of the angle of arrival and the associated correlation function and power spectrum at the mobile station," *IEEE Trans. Vehic. Technol.*, vol. 51, pp. 425-434, 2002.

[20] A. Abdi, "On the second derivative of a Gaussian process envelope," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1226-1231, 2002.

[21] N. M. Blachman, "The distributions of local extrema of Gaussian noise and of its envelope," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2115-2121, 1999.

[22] R. Narasimhan and D. C. Cox, "Speed estimation in wireless systems using wavelets," *IEEE Trans. Commun.*, vol. 47, pp. 1357-1364, 1999.

[23] D. Middleton, "Spurious signals caused by noise in triggered circuits," *J. Appl. Phys.*, vol. 19, pp. 817-830, 1948.

[24] M. R. Leadbetter, "On crossings of levels and curves by a wide class of stochastic processes," *Ann. Math. Statist.*, vol. 37, pp. 260-267, 1966.

[25] A. M. Hasofer, "The joint distribution of the envelope of a Gaussian process and its derivative," *Austral. J. Statist.*, vol. 15, pp. 215-216, 1973.

[26] A. Abdi and M. Kaveh, "Level crossing rate in terms of the characteristic function: A new approach for calculating the fading rate in diversity systems," *IEEE Trans. Commun.*, vol. 50, pp. 1397-1400, 2002.

[27] T. Vinje, "On the statistical distribution of second-order forces and motion," *Int. Shipbulid. Prog.*, vol.30, pp. 58-68, 1983.

[28] V. V. Petrov, *Sums of Independent Random Variables*. Berlin: Springer, 1975.

[29] S. Nader-Esfahani and A. Abdi, "A new formula for the mean number of the peaks of a normal process envelope, in *Proc. Iranian Conf. Elec. Eng.*, Tarbiat Modarres University, Tehran, Iran, 1994, vol. 5, pp. 450-456 (in Persian).

[30] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, corrected and enlarged ed., A. Jeffrey, Ed., New York: Academic, 1980.

[31] J. Dugundji, "Envelopes and pre-envelopes of real waveforms," *IRE Trans. Inform. Theory*, vol. 4, pp. 53-57, 1958.

[32] W. C. Giffin, *Transform Techniques for Probability Modeling*. New York: Academic, 1975.

[33] A. Erdelyi, Ed., *Tables of Integral Transforms*. vol. 1, New York: McGraw-Hill, 1954.

[34] F. Oberhettinger, *Tables of Mellin Transforms.* Berlin: Springer, 1974.

[35] A. Abdi and M. Kaveh, "A new velocity estimator for cellular systems based on higher order crossings," in *Proc. Asilomar Conf. Signals, Systems, Computers*, Pacific Grove, CA, 1998, pp. 1423-1427.

[36] R.N. Bhattacharya and R. R. Rao, *Normal Approximation and Asymptotic Expansions*. New York: Wiley, 1976.

[37] J. E. Mazo and J. Salz, "A theorem on conditional expectation," *IEEE Trans. Inform. Theory*, vol. 16, pp. 379-381, 1970.

[38] B. L. S. Prakasa Rao, "Characterization of stationary processes differentiable in mean square," *IEEE Trans. Inform. Theory*, vol. 18, pp. 659-661, 1972.

[39] S. Lipschutz, *Theory and Problems of Linear Algebra*, 2nd ed., Singapore: McGraw-Hill, 1974.

[40] N. L. Johnson and S. Kotz, *Distributions in Statistics: Continuous Univariate Distributions*. vol. 2, New York: Wiley, 1970.

[41] G. L. Turin, “The characteristic function of Hermitian quadratic forms in complex normal variables,” *Biometrika*, vol. 47, pp. 199-201, 1960.

[42] M. Schwartz, W. R. Bennett, and S. Stein, *Communication Systems and Techniques*. New York: McGraw-Hill, 1966.

[43] S. Wolfram, *Mathematica: A System for Doing Mathematics by Computer*, 2nd ed., Redwood City, CA: Addison-Wesley, 1991.

[44] B. Fristedt and L. Gray, *A Modern Approach to Probability Theory*. Boston, MA: Birkhauser, 1997.

[45] S. O. Rice, “Statistical properties of a sine wave plus random noise,” *Bell Syst. Tech. J.*, vol. 27, pp. 109-157, 1948.

[46] W. B. Davenport, Jr., *Probability and Random Processes*: *An Introduction for Applied Scientists and Engineers*. New York: McGraw-Hill, 1970.

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