

JOINT SINGULAR VALUE DISTRIBUTION OF TWO CORRELATED RECTANGULAR GAUSSIAN MATRICES AND ITS APPLICATION

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Abstract. Let $\mathbf{H} = (h_{ij})$ and $\mathbf{G} = (g_{ij})$ be two $m \times n$, $m \leq n$, *rectangular* random matrices, each with i.i.d complex zero-mean unit-variance Gaussian entries, with correlation between any two elements given by $\mathbb{E}[h_{ij}g_{pq}^*] = \rho \delta_{ip}\delta_{jq}$ such that $|\rho| < 1$, where $*$ denotes the complex conjugate and δ_{ij} is the Kronecker delta. Assume $\{s_k\}_{k=1}^m$ and $\{r_l\}_{l=1}^m$ are unordered singular values of \mathbf{H} and \mathbf{G} , respectively, and s and r are randomly selected from $\{s_k\}_{k=1}^m$ and $\{r_l\}_{l=1}^m$, respectively. In this paper, exact analytical closed-form expressions are derived for the joint probability distribution function (PDF) of $\{s_k\}_{k=1}^m$ and $\{r_l\}_{l=1}^m$ using an Itzykson-Zuber-type integral, as well as the joint marginal PDF of s and r , by a bi-orthogonal polynomial technique. These PDFs are of interest in multiple-input multiple-output (MIMO) wireless communication channels and systems.

Key words. correlated complex random matrices, joint singular value distribution, bi-orthogonal polynomials

AMS subject classifications. 15A52, 15A18, 62E15, 33C45

1. Introduction. Random singular values have found numerous applications such as hypothesis testing and principal component analysis in statistics [14], nuclear energy levels and level spacing in nuclear physics [12], quantum chromodynamics (QCD) [20], and calculation of the multiple-input multiple-output (MIMO) channel capacity in wireless communications [18]. The singular value distribution of a *single* Gaussian random matrix is given in [16]. For a single chiral random matrix, where a square Gaussian random matrix and its Hermitian sit on the off diagonal, the result is reported in [6]. However, the joint singular value distribution of *correlated rectangular* Gaussian random matrices have received less attention so far, although it has important applications in wireless MIMO communications, say, the second-order statistics of the *eigen*-channels [22, Chap. 4] [25] and instantaneous mutual information [22, Chaps. 3 & 4] [23, 24, 26].

To the best of our knowledge, correlated square random matrices have been studied to some extent [4, 12, 13], where only Hermitian matrices were considered. Different from [4, 12, 13], we consider the situation where the elements, with the same indices, of the two rectangular complex Gaussian random matrices are correlated by a *complex* number, and derive exact analytical closed-form expressions for the joint PDF of their singular values.

This paper is organized as follows. Section 2 introduces the two rectangular complex Gaussian random matrices. The joint PDF of singular values are studied in section 3 using an Itzykson-Zuber-type integral. The joint marginal PDF of singular values is derived in section 4 and its application to wireless MIMO communications is presented in section 5. Finally, concluding remarks are summarized in section 6.

Notation: \cdot^\dagger is reserved for matrix Hermitian, \cdot^T for matrix transpose, \cdot^* for complex conjugate, $\text{tr}[\cdot]$ for the trace of a matrix, j for $\sqrt{-1}$, $\mathbb{E}[\cdot]$ for mathematical expectation, \mathbf{I}_m for the $m \times m$ identity matrix, \otimes for the Kronecker product, and $\Re[\cdot]$ and $\Im[\cdot]$ for the real and imaginary parts of a complex number, respectively. In

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addition, $\text{diag}(\mathbf{s})$ denotes a diagonal matrix with \mathbf{s} on the main diagonal, $t \in [m, n]$ implies that t , m , and n are all integers such that $m \leq t \leq n$ with $m \leq n$, and $\det |x_{kl}|$ is the determinant of the matrix, where x_{kl} resides on the k^{th} row and l^{th} column. Moreover, lower-case bold letters represent row vectors, whereas upper-case bold letters are used for matrices. Finally \mathcal{CN} means complex normal, and $\text{vec}(\cdot)$ stacks all the columns of its matrix argument into one tall column vector.

2. Problem Description. There are two $m \times n$ random matrices $\mathbf{H} = (h_{ij})$ and $\mathbf{G} = (g_{ij})$, $i \in [1, m]$, $j \in [1, n]$, each with i.i.d complex zero-mean unit-variance Gaussian entries, i.e., $\mathbb{E}[h_{ij}] = \mathbb{E}[g_{ij}] = 0, \forall i, j$, $\mathbb{E}[h_{ij}h_{pq}^*] = \mathbb{E}[g_{ij}g_{pq}^*] = \delta_{ip}\delta_{jq}$, where the Kronecker symbol δ_{ij} is 1 or 0 when $i = j$ or $i \neq j$. Therefore $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$. Moreover, the correlation among the two random matrices is given by

$$\mathbb{E}[h_{ij}g_{pq}^*] = \rho \delta_{ip}\delta_{jq}, \quad \forall i, j, p, q, \quad (2.1)$$

where $\rho = |\rho|e^{j\theta}$ is a complex number with $|\rho| < 1$.

Without loss of generality, we assume $m \leq n$ and set $\nu = n - m$. Based on the singular value decomposition (SVD), \mathbf{H} and \mathbf{G} can be, respectively, diagonalized as [8]

$$\mathbf{H} = \mathbf{U}\mathbf{S}\mathbf{V}^\dagger, \quad (2.2)$$

$$\mathbf{G} = \tilde{\mathbf{U}}\mathbf{R}\tilde{\mathbf{V}}^\dagger, \quad (2.3)$$

where $\mathbf{S} = [\text{diag}(\mathbf{s}) \mathbf{0}]$ and $\mathbf{R} = [\text{diag}(\mathbf{r}) \mathbf{0}]$ with $\mathbf{s} = [s_1, s_2, \dots, s_m]$ and $\mathbf{r} = [r_1, r_2, \dots, r_m]$, respectively.

We assume that the singular values of \mathbf{G} , r_1, r_2, \dots, r_m , are unordered and the singular values of \mathbf{H} , s_1, s_2, \dots, s_m , are also unordered. Now we would like to know the joint PDF of $\{r_l\}_{l=1}^m$ and $\{s_l\}_{l=1}^m$. Moreover, with r randomly selected from r_1, r_2, \dots, r_m , and s randomly selected from s_1, s_2, \dots, s_m , it is of interest to derive the joint PDF of r and s as well. These two PDFs are derived in Section 3 and 4, respectively.

3. Joint PDF of $\{s_l\}_{l=1}^m$ and $\{r_l\}_{l=1}^m$. LEMMA 3.1 (Joint PDF of \mathbf{H} and \mathbf{G}). *For two correlated rectangular complex Gaussian random matrices, $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$, with the correlation between \mathbf{H} and \mathbf{G} given by (2.1), the joint PDF of \mathbf{H} and \mathbf{G} is given by*

$$p(\mathbf{H}, \mathbf{G}) = \frac{1}{\pi^{2mn} (1 - |\rho|^2)^{mn}} \exp \left[-\frac{\text{tr}(\mathbf{H}\mathbf{H}^\dagger + \mathbf{G}\mathbf{G}^\dagger - \rho^* \mathbf{H}\mathbf{G}^\dagger - \rho \mathbf{G}\mathbf{H}^\dagger)}{1 - |\rho|^2} \right]. \quad (3.1)$$

Proof. We set $\mathbf{h} = \text{vec}(\mathbf{H})$, $\mathbf{g} = \text{vec}(\mathbf{G})$, and $\mathbf{x} = [\mathbf{h}^T \ \mathbf{g}^T]^T$. Based on $\mathbf{H}, \mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{mn})$ and (2.1), we have the mean and covariance matrix of \mathbf{x} as $\mathbb{E}[\mathbf{x}] = \mathbf{0}$ and $\Sigma_{\mathbf{x}} = \Sigma_{\tau} \otimes \mathbf{I}_{mn}$ with $\Sigma_{\tau} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix}$, respectively. Therefore the PDF of \mathbf{x} is given by [11]

$$p(\mathbf{x}) = \frac{1}{\pi^{2mn} \det |\Sigma_{\mathbf{x}}|} \exp(-\mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x}), \quad (3.2)$$

where $\det |\Sigma_{\mathbf{x}}| = (\det |\Sigma_{\tau}|)^{mn} = (1 - |\rho|^2)^{mn}$.

With $\Sigma_\tau^{-1} = \frac{1}{1-|\rho|^2} \begin{bmatrix} 1 & -\rho \\ -\rho^* & 1 \end{bmatrix}$, we obtain $\Sigma_{\mathbf{x}}^{-1} = \Sigma_\tau^{-1} \otimes \mathbf{I}_{mn} = \frac{1}{1-|\rho|^2} \begin{bmatrix} \mathbf{I}_{mn} & -\rho \mathbf{I}_{mn} \\ -\rho^* \mathbf{I}_{mn} & \mathbf{I}_{mn} \end{bmatrix}$. Therefore $\mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x}$ in (3.2) can be rewritten as

$$\begin{aligned} \mathbf{x}^\dagger \Sigma_{\mathbf{x}}^{-1} \mathbf{x} &= \text{tr}(\Sigma_{\mathbf{x}}^{-1} \mathbf{x} \mathbf{x}^\dagger) = \text{tr} \left(\frac{1}{1-|\rho|^2} \begin{bmatrix} \mathbf{I}_{mn} & -\rho \mathbf{I}_{mn} \\ -\rho^* \mathbf{I}_{mn} & \mathbf{I}_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{h} \mathbf{h}^\dagger & \mathbf{h} \mathbf{g}^\dagger \\ \mathbf{g} \mathbf{h}^\dagger & \mathbf{g} \mathbf{g}^\dagger \end{bmatrix} \right), \\ &= \frac{\text{tr}(\mathbf{h} \mathbf{h}^\dagger + \mathbf{g} \mathbf{g}^\dagger - \rho^* \mathbf{h} \mathbf{g}^\dagger - \rho \mathbf{g} \mathbf{h}^\dagger)}{1-|\rho|^2} = \frac{\text{tr}(\mathbf{H} \mathbf{H}^\dagger + \mathbf{G} \mathbf{G}^\dagger - \rho^* \mathbf{H} \mathbf{G}^\dagger - \rho \mathbf{G} \mathbf{H}^\dagger)}{1-|\rho|^2}, \end{aligned} \quad (3.3)$$

where $\text{tr}(\mathbf{A} \mathbf{B}^\dagger) = \text{vec}(\mathbf{B})^\dagger \text{vec}(\mathbf{A}) = \text{tr}[\text{vec}(\mathbf{A}) \text{vec}(\mathbf{B})^\dagger]$ [7] is used in the last “=” of (3.3). Substitution of (3.3) into (3.2) leads to (3.1). \square

From (2.2), we know that the unitary matrix pair (\mathbf{U}, \mathbf{V}) parameterizes the coset space $\mathcal{U}(m) \times \mathcal{U}(n) / [\mathcal{U}(1)]^m$, where $\mathcal{U}(p)$ is the unitary group of order p , and the integration measure, $d[\mathbf{H}] = \prod_{i=1}^m \prod_{j=1}^n d[\Re h_{ij}] d[\Im h_{ij}]$, can be represented by [9]

$$d[\mathbf{H}] = \Omega J(\mathbf{s}) d[\mathbf{s}] d\mu(\mathbf{U}, \mathbf{V}), \quad (3.4)$$

where $J(\mathbf{s}) = \Delta^2(\mathbf{s}^2) \prod_{k=1}^m s_k^{2\nu+1}$ with the m -dimensional Vandermonde determinant $\Delta(\mathbf{s}^2) = \det[s_k^{2(l-1)}] = \prod_{k>l} (s_k^2 - s_l^2)$ and $\Delta^2(\cdot) = [\Delta(\cdot)]^2$, $d[\mathbf{s}] = \prod_{l=1}^m ds_l$, $d\mu(\mathbf{U}, \mathbf{V})$ is the Haar measure of $\mathcal{U}(m) \times \mathcal{U}(n) / [\mathcal{U}(1)]^m$ [9], and the constant Ω is given by [9, 15]

$$\Omega = \frac{2^m \pi^{mn}}{\prod_{j=1}^m j!(j+\nu-1)!} = \frac{2^m \pi^{mn}}{m! \prod_{j=0}^{m-1} j!(j+\nu)!}. \quad (3.5)$$

Similarly, we have

$$d[\mathbf{G}] = \Omega J(\mathbf{r}) d[\mathbf{r}] d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}), \quad (3.6)$$

where $J(\mathbf{r}) = \Delta^2(\mathbf{r}^2) \prod_{k=1}^m r_k^{2\nu+1}$ with the m -dimensional Vandermonde determinant $\Delta(\mathbf{r}^2) = \det[r_k^{2(l-1)}] = \prod_{k>l} (r_k^2 - r_l^2)$ and $d[\mathbf{r}] = \prod_{l=1}^m dr_l$.

In order to obtain the joint PDF of $\{r_l\}_{l=1}^m$ and $\{s_l\}_{l=1}^m$, we need the following proposition.

PROPOSITION 3.2 (An Itzykson-Zuber-type integral [9, (31)]).

$$\begin{aligned} &\int d\mu(\mathbf{U}, \mathbf{V}) \exp \left\{ -\frac{\text{tr}[(\mathbf{H} - \mathbf{G})(\mathbf{H} - \mathbf{G})^\dagger]}{t} \right\} \\ &= \frac{2^m \pi^{mn} t^{mn-m} \det \left| \exp \left(-\frac{s_k^2 + r_l^2}{t} \right) I_\nu \left(\frac{2s_k r_l}{t} \right) \right|}{m! \Omega \Delta(\mathbf{s}^2) \Delta(\mathbf{r}^2) \prod_{k=1}^m (s_k r_k)^\nu}, \end{aligned} \quad (3.7)$$

where Ω is given by (3.5) and $I_k(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(k\theta) d\theta$ is the k^{th} order modified Bessel function of the first kind.

THEOREM 3.3. The joint PDF of the singular values of \mathbf{H} and \mathbf{G} is given by

$$p(\mathbf{s}, \mathbf{r}) = \frac{\exp \left(-\frac{\sum_{k=1}^m \frac{s_k^2 + r_k^2}{1-|\rho|^2}}{1-|\rho|^2} \right) \Delta(\mathbf{s}^2) \Delta(\mathbf{r}^2) \prod_{k=1}^m (s_k r_k)^{\nu+1} \det \left| I_\nu \left(\frac{2|\rho| s_k r_l}{1-|\rho|^2} \right) \right|}{2^{-2m} m! m! \prod_{j=0}^{m-1} j!(j+\nu)! |\rho|^{mn-m} (1-|\rho|^2)^m}. \quad (3.8)$$

Proof. By combining (3.1) with (3.4) and (3.6), we obtain

$$p(\mathbf{s}, \mathbf{r}) = \frac{\Omega^2 J(\mathbf{s}) J(\mathbf{r})}{\pi^{2mn} (1 - |\rho|^2)^{mn}} \Phi(\mathbf{s}, \mathbf{r}), \quad (3.9)$$

where

$$\begin{aligned} \Phi(\mathbf{s}, \mathbf{r}) &= \int d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) \int d\mu(\mathbf{U}, \mathbf{V}) \exp \left[-\frac{\text{tr}(\mathbf{H}\mathbf{H}^\dagger + \mathbf{G}\mathbf{G}^\dagger - \rho^* \mathbf{H}\mathbf{G}^\dagger - \rho \mathbf{G}\mathbf{H}^\dagger)}{1 - |\rho|^2} \right], \\ &= \int d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) \int d\mu(\mathbf{U}, \mathbf{V}) \exp \left\{ -\frac{\text{tr}[(\mathbf{H} - \rho \mathbf{G})(\mathbf{H} - \rho \mathbf{G})^\dagger]}{1 - |\rho|^2} - \text{tr}(\mathbf{G}\mathbf{G}^\dagger) \right\}, \\ &= \int d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) e^{-\text{tr}(\mathbf{G}\mathbf{G}^\dagger)} \int d\mu(\mathbf{U}, \mathbf{V}) \exp \left\{ -\frac{\text{tr}[(\mathbf{H} - \rho \mathbf{G})(\mathbf{H} - \rho \mathbf{G})^\dagger]}{1 - |\rho|^2} \right\}, \\ &= \int d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) \frac{e^{-\sum_{k=1}^m r_k^2 (1 - |\rho|^2)^{mn-m}} \det \left| e^{-\frac{s_k^2 + |\rho|^2 r_l^2}{1 - |\rho|^2}} I_\nu \left(\frac{2|\rho| s_k r_l}{1 - |\rho|^2} \right) \right|}{2^{-m} m! \pi^{-mn} \Omega \Delta(\mathbf{s}^2) \Delta(|\rho|^2 \mathbf{r}^2) \prod_{k=1}^m (|\rho| s_k r_k)^\nu}, \\ &= \frac{(1 - |\rho|^2)^{mn-m} \exp \left(-\frac{\sum_{k=1}^m s_k^2 + r_k^2}{1 - |\rho|^2} \right) \det \left| I_\nu \left(\frac{2|\rho| s_k r_l}{1 - |\rho|^2} \right) \right|}{2^{-m} m! \pi^{-mn} \Omega |\rho|^{mn-m} \Delta(\mathbf{s}^2) \Delta(\mathbf{r}^2) \prod_{k=1}^m (s_k r_k)^\nu}. \end{aligned} \quad (3.10)$$

Derivation of the second and third lines of (3.10) are straightforward. The fourth line comes from

$$\rho \mathbf{G} = \hat{\mathbf{U}} \hat{\mathbf{R}} \hat{\mathbf{V}}^\dagger \quad (3.11)$$

with $\hat{\mathbf{R}} = |\rho| \mathbf{R}$, and Proposition 3.2 with the replacements $t \rightarrow 1 - |\rho|^2$ and $\mathbf{G} \rightarrow \rho \mathbf{G}$. The last line is based on the convention that $\int d\mu(\tilde{\mathbf{U}}, \tilde{\mathbf{V}}) = 1$ [9]. Plugging (3.5) and the last line of (3.10) into (3.9), we obtain (3.8). \square

By relating the eigenvalues of $\mathbf{G}\mathbf{G}^\dagger$ to the singular values of \mathbf{G} through $\alpha_l = r_l^2$, $l \in [1, m]$, and the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ to the singular values of \mathbf{H} through $\beta_l = s_l^2$, $l \in [1, m]$, we can derive the joint PDF of $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_m]$ and $\boldsymbol{\beta} = [\beta_1, \beta_2, \dots, \beta_m]$, presented in the following corollary.

COROLLARY 3.4. *The joint PDF of the unordered eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ and $\mathbf{G}\mathbf{G}^\dagger$ is*

$$p(\boldsymbol{\beta}, \boldsymbol{\alpha}) = \frac{\exp \left(-\frac{\sum_{k=1}^m \beta_k + \alpha_k}{1 - |\rho|^2} \right) \Delta(\boldsymbol{\beta}) \Delta(\boldsymbol{\alpha}) \prod_{k=1}^m (\sqrt{\beta_k \alpha_k})^\nu \det \left| I_\nu \left(\frac{2|\rho| \sqrt{\beta_k \alpha_l}}{1 - |\rho|^2} \right) \right|}{m! m! \prod_{j=0}^{m-1} j! (j + \nu)! |\rho|^{mn-m} (1 - |\rho|^2)^m}, \quad (3.12)$$

where m -dimensional Vandermonde determinants are defined by $\Delta(\boldsymbol{\beta}) = \det |\beta_k^{l-1}| = \prod_{k>l} (\beta_k - \beta_l)$ and $\Delta(\boldsymbol{\alpha}) = \det |\alpha_k^{l-1}| = \prod_{k>l} (\alpha_k - \alpha_l)$.

Proof. It is straightforward to obtain (3.12) from (3.8) by $2m$ one-to-one nonlinear mappings. \square

4. Joint Marginal PDF. In this section, with $\beta = s^2$ and $\alpha = r^2$, we calculate the joint marginal PDF of β and α , $p(\beta, \alpha)$, using the techniques and results presented in [4, 13]. Then the joint PDF of s and r , $p(s, r)$, is easily derived.

If the polynomials $P_k(\beta)$ and $Q_l(\alpha)$, satisfy $\int w(\beta, \alpha) P_k(\beta) Q_l(\alpha) d\beta d\alpha = \delta_{kl}$, then we call $P_k(\beta)$ and $Q_l(\alpha)$ as bi-orthogonal polynomials, associated with the weight function $w(\beta, \alpha)$ [12]. With this definition, we have the following Lemma.

LEMMA 4.1. *There exist bi-polynomials, $P_k(\beta)$ and $Q_l(\alpha)$, and a weight function, $w(\beta, \alpha)$, which reduce (3.12) to the following form*

$$p(\beta, \alpha) = C_1 \det |P_{k-1}(\beta_l)| \det |w(\beta_k, \alpha_l)| \det |Q_{k-1}(\alpha_l)|, \quad (4.1)$$

where C_1 is a normalization constant.

Proof. In this paper, ν is a non-negative integer. Using the Hille-Hardy formula [2, pp. 185, (46)]

$$\sum_{k=0}^{\infty} \frac{k! z^k}{(k + \nu)!} L_k^\nu(x) L_k^\nu(y) = \frac{(xyz)^{-\frac{\nu}{2}}}{1 - z} \exp\left(-z \frac{x + y}{1 - z}\right) I_\nu\left(\frac{2\sqrt{xyz}}{1 - z}\right), |z| < 1, \quad (4.2)$$

with $L_k^\nu(x) = \frac{1}{k!} e^x x^{-\nu} \frac{d^k}{dx^k} (e^{-x} x^{k+\nu})$ as the associated Laguerre polynomial, we can rewrite (3.12) as

$$p(\beta, \alpha) = \frac{\Delta(\beta) \Delta(\alpha) \det \left| \beta_k^\nu e^{-\beta_k} \alpha_l^\nu e^{-\alpha_l} \sum_{j=0}^{\infty} \frac{j! |\rho|^{2j} L_j^\nu(\beta_k) L_j^\nu(\alpha_l)}{(j + \nu)!} \right|}{m! m! \prod_{j=0}^{m-1} j! (j + \nu)! |\rho|^{m(m-1)}}. \quad (4.3)$$

We set the weight function, $w(\beta, \alpha)$, as

$$\begin{aligned} w(\beta, \alpha) &= \beta^\nu \alpha^\nu e^{-(\beta + \alpha)} \sum_{j=0}^{\infty} \frac{j! |\rho|^{2j} L_j^\nu(\beta) L_j^\nu(\alpha)}{(j + \nu)!}, \\ &= \frac{(\beta \alpha)^{\frac{\nu}{2}} e^{-\frac{\beta + \alpha}{1 - |\rho|^2}} I_\nu\left(\frac{2|\rho| \sqrt{\beta \alpha}}{1 - |\rho|^2}\right)}{(1 - |\rho|^2) |\rho|^\nu}. \end{aligned} \quad (4.4)$$

It is easy to check that the corresponding bi-orthogonal polynomials are given by

$$P_k(\beta) = \sqrt{\frac{k!}{(k + \nu)!}} |\rho|^{-k} L_k^\nu(\beta), \quad (4.5)$$

$$Q_l(\alpha) = \sqrt{\frac{l!}{(l + \nu)!}} |\rho|^{-l} L_l^\nu(\alpha), \quad (4.6)$$

using the following integral equality [2, pp. 267, 7.414.3]

$$\int_0^\infty e^{-x} x^\nu L_k^\nu(x) L_l^\nu(x) dx = \frac{(k + \nu)!}{k!} \delta_{kl} \quad (4.7)$$

Moreover, by the addition of multiples of rows of lower order which do not change the determinant of the Vandermonde matrix, then each of the rows can be expressed in terms of orthogonal polynomials with respect to the weight function $w(\beta, \alpha)$. Therefore two m -dimensional Vandermonde determinants, $\Delta(\beta)$ and $\Delta(\alpha)$, can be represented as

$$\Delta(\beta) = \det |\beta_k^{l-1}| = C_2 \det |P_{k-1}(\beta_l)|, \quad (4.8)$$

$$\Delta(\alpha) = \det |\alpha_k^{l-1}| = C_3 \det |Q_{k-1}(\alpha_l)|, \quad (4.9)$$

where we use the fact that the matrix transpose does not change the determinant, i.e., $\det |P_{l-1}(\beta_k)| = \det |P_{k-1}(\beta_l)|$ and $\det |Q_{l-1}(\beta_k)| = \det |Q_{k-1}(\beta_l)|$.

The coefficient of x^k in $L_k^\nu(x)$ is $\frac{(-1)^k}{k!}$, then the coefficient of x^k in $P_k(x)$ is $(-1)^k |\rho|^{-k} \frac{1}{\sqrt{k!(k+\nu)!}}$, therefore we have $C_2 = \prod_{j=0}^{m-1} (-1)^j |\rho|^j \sqrt{j!(j+\nu)!} = (-1)^{\frac{m(m-1)}{2}} \times \sqrt{|\rho|^{m(m-1)} \prod_{j=0}^{m-1} j!(j+\nu)!}$, obtained by plugging (4.5) into (4.8). Similarly, substitution of (4.6) into (4.9) gives $C_3 = C_2$. Now the product of (4.8) and (4.9) results in

$$\Delta(\beta)\Delta(\alpha) = |\rho|^{m(m-1)} \prod_{j=0}^{m-1} j!(j+\nu)! \det |P_{k-1}(\beta_l)| \det |Q_{k-1}(\alpha_l)|. \quad (4.10)$$

Based on (4.4) and (4.10), one can see that (4.3) is equal to (4.1) with $C_1 = \frac{1}{m!m!}$.

□

THEOREM 4.2. *The joint PDF of β and α is given by*

$$\begin{aligned} p(\beta, \alpha) = & \frac{(\beta\alpha)^{\frac{\nu}{2}} e^{-\frac{\beta+\alpha}{1-|\rho|^2}} I_\nu\left(\frac{2|\rho|\sqrt{\beta\alpha}}{1-|\rho|^2}\right)}{m^2(1-|\rho|^2)|\rho|^\nu} \sum_{k=0}^{m-1} \frac{k!}{(k+\nu)!} \frac{L_k^\nu(\beta)L_k^\nu(\alpha)}{|\rho|^{2k}} \\ & + \frac{(\beta\alpha)^\nu e^{-(\beta+\alpha)}}{m^2} \sum_{0 \leq k < l}^{m-1} \left\{ \frac{k!l!}{(k+\nu)!(l+\nu)!} \left\{ [L_k^\nu(\beta)L_l^\nu(\alpha)]^2 + [L_l^\nu(\beta)L_k^\nu(\alpha)]^2 \right. \right. \\ & \left. \left. - \left[|\rho|^{2(l-k)} + |\rho|^{2(k-l)} \right] L_k^\nu(\beta)L_l^\nu(\beta)L_k^\nu(\alpha)L_l^\nu(\alpha) \right\} \right\}. \quad (4.11) \end{aligned}$$

Proof. Based on Lemma 4.1, and the results presented in [13, (3.7)] [4], $p(\beta, \alpha)$ can be expressed as

$$m^2 p(\beta, \alpha) = w(\beta, \alpha) \sum_{k=0}^{m-1} P_k(\beta)Q_k(\alpha) + \sum_{0 \leq k < l}^{m-1} \det \begin{vmatrix} P_k(\beta) & \bar{P}_k(\alpha) \\ P_l(\beta) & \bar{P}_l(\alpha) \end{vmatrix} \det \begin{vmatrix} \bar{Q}_k(\beta) & Q_k(\alpha) \\ \bar{Q}_l(\beta) & Q_l(\alpha) \end{vmatrix}, \quad (4.12)$$

where $P_k(x)$ and $Q_k(x)$ are defined in (4.5) and (4.6), respectively, the weight function is presented in (4.4), and $\bar{P}_k(\alpha)$ and $\bar{Q}_l(\beta)$ are similarly defined as [13]

$$\bar{P}_k(\alpha) = \int P_k(\beta)w(\beta, \alpha)d\beta = \sqrt{\frac{k!}{(k+\nu)!}} \alpha^\nu e^{-\alpha} |\rho|^k L_k^\nu(\alpha), \quad (4.13)$$

$$\bar{Q}_l(\beta) = \int Q_l(\alpha)w(\beta, \alpha)d\alpha = \sqrt{\frac{l!}{(l+\nu)!}} \beta^\nu e^{-\beta} |\rho|^l L_l^\nu(\beta). \quad (4.14)$$

Plugging (4.4), (4.5), (4.6), (4.13) and (4.14) into (4.12), we arrive at (4.11). □

It is straightforward to obtain the joint PDF of s and r from (4.11), according to these one-to-one mappings $s = \sqrt{\beta}$ and $r = \sqrt{\alpha}$.

The joint PDF in (4.11) includes many existing PDF's as special cases.

- By integration over β , (4.11) reduces to the marginal PDF

$$p(\alpha) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{k!}{(k+\nu)!} [L_k^\nu(\alpha)]^2 \alpha^\nu e^{-\alpha}, \quad (4.15)$$

which is the same as the PDF presented in [18]. When $m = 1$, (4.15) further reduces to

$$p(\alpha) = \frac{1}{(n-1)!} \alpha^{n-1} e^{-\alpha}, \quad (4.16)$$

which is the χ^2 distribution with $2n$ degrees of freedom [17, (2.32)].

- With $m = 1$, (4.11) reduces to [21],

$$p(\alpha, \beta) = \frac{(\alpha\beta)^{\frac{n-1}{2}} \exp\left(-\frac{\alpha+\beta}{1-|\rho|^2}\right) I_{n-1}\left(\frac{2|\rho|\sqrt{\alpha\beta}}{1-|\rho|^2}\right)}{(n-1)! (1-|\rho|^2) |\rho|^{n-1}}. \quad (4.17)$$

Furthermore, when $n = 1$, (4.17) simplifies to

$$p(\alpha, \beta) = \frac{1}{1-|\rho|^2} \exp\left(-\frac{\alpha+\beta}{1-|\rho|^2}\right) I_0\left(\frac{2|\rho|\sqrt{\alpha\beta}}{1-|\rho|^2}\right), \quad (4.18)$$

which is identical to (8-103) [3, pp. 163], after two one-to-one nonlinear transformations.

For the application discussed in section 5, we need the joint marginal PDF of ϕ and φ , $p(\phi, \varphi)$, where ϕ and φ are randomly selected from $\{\alpha_k\}_{k=1}^m$, $m \geq 2$. Using the technique in [4, 13], we have the following theorem.

THEOREM 4.3. *If ϕ and φ are randomly selected from $\{\alpha_k\}_{k=1}^m$, their joint PDF is given by*

$$p(\phi, \varphi) = \frac{(\phi\varphi)^\nu e^{-(\phi+\varphi)}}{m(m-1)} \sum_{\substack{k,l=0 \\ k \neq l}}^{m-1} \frac{k!l!}{(k+\nu)!(l+\nu)!} \left\{ [L_k^\nu(\phi) L_l^\nu(\varphi)]^2 - L_k^\nu(\phi) L_l^\nu(\phi) L_k^\nu(\varphi) L_l^\nu(\varphi) \right\}. \quad (4.19)$$

Proof. According to (1.6) and (2.14) in [13] we have

$$p(\phi, \varphi) = \frac{1}{m(m-1)} \det \begin{vmatrix} K(\phi, \phi) & K(\phi, \varphi) \\ K(\varphi, \phi) & K(\varphi, \varphi) \end{vmatrix}, \quad (4.20)$$

where $K(x_1, x_2) = \sum_{k=0}^{m-1} P_k(x_1) \overline{Q}_k(x_2)$. With $P_k(x_1)$ in (4.5) and $\overline{Q}_k(x_2)$ in (4.14), we obtain (4.19) after some simple algebraic manipulations. \square

5. Application to Wireless MIMO Communication Systems. For an $N_R \times N_T$ MIMO time-varying Rayleigh flat fading channel [19] with N_T transmitters and N_R receivers, the channel impulse response at time instant t is given by

$$\mathbf{H}(t) = \begin{bmatrix} h_{1,1}(t) & \cdots & h_{1,N_T}(t) \\ \vdots & \ddots & \vdots \\ h_{N_R,1}(t) & \cdots & h_{N_R,N_T}(t) \end{bmatrix}. \quad (5.1)$$

We assume all the $N_R N_T$ sub-channels in the MIMO system, $\{h_{i,j}(t)\}_{(i=1,j=1)}^{(N_R,N_T)}$ are i.i.d., with the same temporal correlation coefficient, i.e.,

$$\mathbb{E}[h_{ij}(t) h_{pq}^*(t - \tau)] = \delta_{ip} \delta_{jq} \rho_h(\tau), \quad (5.2)$$

where $\rho_h(\tau) = J_0(2\pi f_D \tau)$ [10] in isotropic scattering environments¹, with $J_0(x) = I_0(-jx)$ [5, pp. 961, 8.406.3] and f_D is the maximum Doppler frequency shift.

We set $n = \max(N_R, N_T)$ and $m = \min(N_R, N_T)$. According to (2.2), $\mathbf{H}(t)$ can be diagonalized as

$$\mathbf{H}(t) = \mathbf{U}(t)\mathbf{S}(t)\mathbf{V}^\dagger(t), \quad (5.3)$$

where $\mathbf{S}(t) = [\text{diag}(\mathbf{s}(t)) \mathbf{0}]$ with $\mathbf{s}(t) = [s_1(t), s_2(t), \dots, s_m(t)]$ for $N_R \leq N_T$, and $\mathbf{S}(t) = \begin{bmatrix} \text{diag}(\mathbf{s}(t)) \\ \mathbf{0} \end{bmatrix}$ for $N_R > N_T$. Therefore the MIMO channel, $\mathbf{H}(t)$, is decomposed to m identically distributed *eigen*-channels $\lambda_k(t) = s_k^2(t)$, $k \in [1, m]$, by SVD.

In wireless MIMO communication systems, we are interested in the correlation coefficient between any two *eigen*-channels, which is defined by

$$\rho_{k,l}(\tau) = \frac{\mathbb{E}[\lambda_k(t)\lambda_l(t-\tau)] - \mathbb{E}[\lambda_k(t)]\mathbb{E}[\lambda_l(t)]}{\sqrt{\mathbb{E}[\lambda_k^2(t)] - \{\mathbb{E}[\lambda_k(t)]\}^2} \sqrt{\mathbb{E}[\lambda_l^2(t)] - \{\mathbb{E}[\lambda_l(t)]\}^2}}. \quad (5.4)$$

For simplicity, in this paper we only consider a 2×2 MIMO system², $N_R = N_T = 2$, where the correlation coefficient, $\rho_{k,l}(\tau)$, can be shown to be

$$\rho_{k,l}(\tau) = \begin{cases} 1 - \frac{3}{2}(1 - \delta_{kl}), & \tau = 0, \\ \frac{|\rho_h(\tau)|^2}{4} = \frac{J_0^2(2\pi f_D \tau)}{4}, & \tau \neq 0, \end{cases}, k, l = 1, 2, \quad (5.5)$$

with $J_0^2(\cdot) = [J_0(\cdot)]^2$. To derive (5.5), we note that for $\tau = 0$ and $k = l$, $\rho_{k,l}(0) = 1$ because of the definition of the correlation coefficient. Since $m = 2$, for any *eigen*-channel at the time instant t , it is easy to show that the mean value of $\lambda_k(t)$ is $\mathbb{E}[\lambda_k(t)] = 2$, $\forall k$, and the second moment of $\lambda_k(t)$ is $\mathbb{E}[\lambda_k^2(t)] = 8$, $\forall k$, using the PDF in (4.15). For $\tau = 0$ and $k \neq l$, we obtain $\mathbb{E}[\lambda_k(t)\lambda_l(t)] = 2$ by (4.19), hence $\rho_{k,l}(0) = -\frac{1}{2}$, $\forall k \neq l$. For $\tau \neq 0$ and $\forall k, l$, it is not difficult to get $\mathbb{E}[\lambda_k(t)\lambda_l(t-\tau)] = 4 + |\rho_h(\tau)|^2$ using (4.11), therefore we have the second line in (5.5).

Monte Carlo simulations are performed to verify the result in (5.5). In all simulations³, the maximum Doppler frequency f_D is set to 1 Hz, and the sampling period, T_s , is equal to $\frac{1}{1000f_D}$. The simulation results are shown in FIG. 5.1, where the upper figure shows the channel correlation coefficient $\rho_h(\tau) = J_0(2\pi f_D \tau)$, Clarke's correlation model, whereas the lower figure presents the correlation coefficient between any two *eigen*-channels or for any individual *eigen*-channel, Eq. (5.5). Since $J_0(2\pi f_D \tau)$ is an even function of τ , the correlation coefficients are plotted for $\tau \geq 0$. In all figures, "Simu." indicates the curve is obtained by Monte Carlo simulations, whereas "Theo." means theoretical. From FIG. 5.1 we can conclude that the new theoretical result in (5.5) is confirmed by simulation very well.

6. Conclusion. In this paper, the joint distribution of singular values of two correlated rectangular complex Gaussian random matrices is derived, as well as the joint marginal distribution. The derived distributions play an important role in the analysis and design of wireless MIMO communication systems. As an example, the correlation coefficient of any two *eigen*-channels of a 2×2 MIMO system is obtained and verified by the Monte Carlo simulations in this paper.

¹In the non-isotropic scattering environment, $\rho_h(\tau)$, in general, is a complex-value function [22, 23], and $|\rho_h(\tau)|$ indicates its amplitude at the time delay τ .

²The general $N_R \times N_T$ MIMO system is considered in [22, Chap. 4].

³The spectral method [1] is used to generate the MIMO channels.

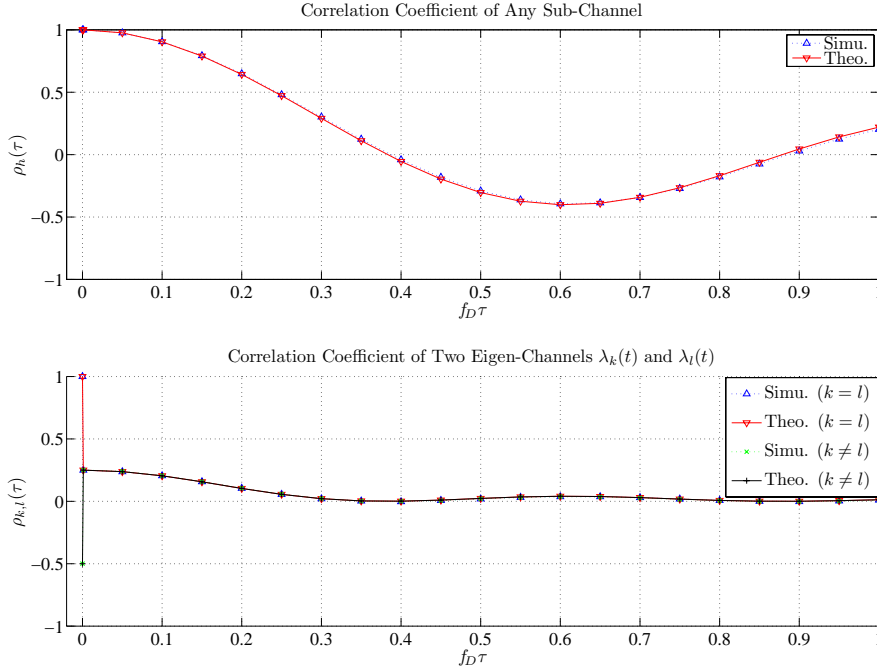


FIG. 5.1. The channel correlation coefficient, $\rho_h(\tau)$, and correlation coefficient of any two eigen-channels, $\rho_{k,l}(\tau)$, in a 2×2 MIMO system with Clarke's correlation model. Note that the sampling period, T_s , is $\frac{1}{1000f_D}$ in Monte Carlo simulations, therefore the first non-zero τ is T_s , i.e., $\frac{1}{1000f_D}$, which corresponds to $f_D\tau = \frac{1}{1000}$ in the horizontal axis.

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REFERENCES

- [1] K. ACOLATSE AND A. ABDI, *Efficient simulation of space-time correlated MIMO mobile fading channels*, in Proc. IEEE Veh. Technol. Conf., Orlando, FL, 2003, pp. 652–656.
- [2] P. BECKMANN, *Orthogonal Polynomials for Engineers and Physicists*, Golem Press, Boulder, CO, 1973.
- [3] W. B. DAVENPORT AND W. L. ROOT, *An Introduction to the Theory of Random Signals and Noise*, Wiley, New York, 1987.
- [4] B. EYNARD AND M. L. MEHTA, *Matrices coupled in a chain: I. eigenvalue correlations*, J. Phys. A: Math. Gen., 31 (1998), pp. 4449–4456.
- [5] I. S. GRADSHTEYN, I. M. RYZHIK, AND A. JEFFREY, eds., *Table of Integrals, Series, and Products*, Academic, San Diego, CA, fifth ed., 1994.
- [6] T. GUHR AND T. WETTIG, *Universal spectral correlations of the Dirac operator at finite temperature*, Nucl. Phys. B, 506 (1997), pp. 589–611.
- [7] A. K. GUPTA AND D. K. NAGAR, *Matrix Variate Distributions*, Chapman & Hall/CRC, New York, 1999.
- [8] L. K. HUA, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domain*, American Mathematical Society, Providence, RI, 1963.
- [9] A. D. JACKSON, M. K. ŞENER, AND J. J. M. VERBAARSCHOT, *Finite volume partition functions and Itzykson-Zuber integrals*, Phys. Lett. B, 387 (1996), pp. 355–360.
- [10] W. C. JAKES, ed., *Microwave Mobile Communications*, IEEE Press, New York, 1994.
- [11] A. T. JAMES, *Distributions of matrix variates and latent roots derived from normal samples*, Ann. Math. Statist., 35 (1964), pp. 475–501.

- [12] M. L. MEHTA, *Random Matrices*, Academic Press, Boston, MA, 2004.
- [13] M. L. MEHTA AND P. SHUKLA, *Two coupled matrices: eigenvalue correlations and spacing functions*, J. Phys. A: Math. Gen., 27 (1994), pp. 7793–7803.
- [14] R. J. MUIRHEAD, *Aspects of Multivariate Statistical Theory*, Wiley, New York, 1982.
- [15] T. NAGAO AND M. WADATI, *Correlation functions of random matrix ensembles related to classical orthogonal polynomials*, J. Phys. Soc. Japan, 60 (1991), pp. 3298–3322.
- [16] J. SHEN, *On the singular values of Gaussian random matrices*, Linear Alg. Appl., 326 (2001), pp. 1–14.
- [17] M. K. SIMON, *Probability Distributions Involving Gaussian Random Variables: A Handbook for Engineers and Scientists*, Kluwer, Boston, MA, 2002.
- [18] İ. E. TELATAR, *Capacity of multi-antenna Gaussian channels*, European Trans. Telecommun., 10 (1999), pp. 585–595.
- [19] D. TSE AND P. VISWANATH, *Fundamentals of Wireless Communication*, Cambridge University Press, Cambridge, UK, 2005.
- [20] J. J. M. VERBAARSCHOT AND T. WETTIG, *Random matrix theory and chiral symmetry in QCD*, Ann. Rev. Nuc. Par. Sci., 50 (2000), pp. 343–410.
- [21] S. WANG, *Envelope correlation coefficient for logarithmic diversity receivers revisited, to be published in IEEE Trans. Commun.*, (2006).
- [22] ———, *MIMO Fading Channels: Models, Statistics, and Low-Complexity Estimators*, PhD thesis, New Jersey Institute of Technology, Newark, NJ, 2006.
- [23] S. WANG AND A. ABDI, *On the second-order statistics of the instantaneous mutual information of time-varying fading channels*, in Proc. IEEE Int. Workshop Signal Processing Advances in Wireless Communications, New York, 2005, pp. 405–409.
- [24] ———, *Correlation analysis of instantaneous mutual information in 2×2 MIMO systems*, in Proc. 40th Ann. Conf. Info. Sci. Sys., Princeton, NJ, 2006, pp. 542–546.
- [25] ———, *Statistical characterization of eigen-channels in time-varying Rayleigh flat fading MIMO systems*, in to be published in Proc. IEEE Global Telecommun. Conf., San Francisco, CA, 2006.
- [26] N. ZHANG AND B. VOJCIC, *Evaluating the temporal correlation of MIMO channel capacities*, in Proc. IEEE Global Telecommun. Conf., St. Louis, MO, 2005, pp. 2817–2821.