A Recursive Approach to Compute Bit Error Rate in Underwater Channels with Multiple Paths

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***Abstract* — In underwater acoustic channels, signal is transmitted over several distinct paths, eigenpaths, from transmitter to receiver due to reflections at sea boundaries, where each eigenpath contains a dominant specular component and a number of scattered components. As a result, an underwater acoustic channel response is the superposition of several dominant specular components and numerous scattered components. Bit error rate (BER) in multipath fading channels has been extensively studied in the past. However, limited research has been conducted on fading channels with several dominant specular components. In this paper, BER in multipath channels with several specular components is studied. A new formula to compute the BER recursively and efficiently is derived. Then using Jensen's inequality, one specular component, Rice fading, is shown to provide the lowest possible BER. Upon using the new BER formula and Lagrange multipliers to solve a constrained optimization problem, it is further shown that for two dominant specular components, BER achieves its maximum when the two components are equally weighted. More results on BER for three and four specular paths are also presented. The results shed light on the impact of the number of specular paths on BER, as well as the maximum and minimum values of BER, which are of interest in underwater communication systems.**

***Index Terms* — Bit error rate, multipath fading channels, underwater channels, acoustic communication, constrained optimization, wireless propagation.**

I. INTRODUCTION

A real underwater acoustic channel contains several distinct paths [1] [2]. Besides the direct path from the transmitter to the receiver, reflections at the sea surface and bottom also create new paths. However, signal travelling along each particular path experiences the inhomogeneity and thermal microstructure of the ocean medium, which turn the received signal into a dominant specular component and a number of randomly scattered components. Upon the superposition of several distinct paths, an underwater acoustic channel response can be considered to include several dominant specular components and many diffuse scattered components. It has been observed through at-sea experiments that an underwater acoustic channel could have two or three or even more specular components [3] [4].

In the past, a large volume of research has been done on the bit error rate (BER) of digital modulations for multipath channels and in the presence of diffuse components, Rayleigh fading, as well as diffuse plus one specular component, Rice fading [5]. However, how the increasing number of specular components may affect the BER is unknown. Therefore, it is of interest to study the BER of fading channels with several specular components, especially in underwater acoustic systems. The probability density function (PDF) of the signal envelope for more than one specular component is discussed in [6]-[8]. The case of two specular components is discussed in [6] and [7]. Two and more specular components are thoroughly studied in [8], where some series for BER and the envelope PDF are derived, with minimal truncation errors.

In this paper, we study this problem from another perspective, by adding the specular components in multiple steps. Therefore, a new recursive formula is derived, which has high computational efficiency in calculating the BER. New results are derived using Jensen's inequality and Lagrange multipliers for constrained optimization, to determine the impact of the number of specular components on BER, and the highest and lowest values of BER.

The rest of the paper is organized as follows. Section II starts with problem formulation and derives the new recursive BER formula, as well as its minima and maxima. Section III provides numerical results on BER in fading channels with multiple specular components. Finally, Section IV provides the concluding remarks and discusses some other possible applications of the derived results and formulas.

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II. Derivation of the Recursive Formula

The received signal in a multipath fading channel with N specular components and a diffuse component can be represented by [6] [8]

$$R\exp(j\Theta)=\sum_{i=1}^{N}a_i\exp(j\Phi_i)+A_0\exp(j\Phi_0), \qquad (1)$$

where R and Θ are the received amplitude and phase, $j^2=-1$, a_i and Φ_i are the amplitude and phase of the i-th specular component, and A_0 and Φ_0 are the amplitude and phase of the diffuse component. The phases Φ_i, $i=0,1,2,\ \ldots,N$ are independent and identically distributed uniform random variables over $[0,2\pi)$, whereas a_i's are constant and A_0 has a Rayleigh distribution with average power $\Omega_0=E[A_0^2]$. Let us define the superposition of specular components as $B_N\exp(j\Psi_N)=\sum_{i=1}^{N}a_i\exp(j\Phi_i)$. Conditioned on B_N, the PDF of R is the Rice pdf [8]

$$f_{R|B_N}(r\mid b_N)=(2\Omega_0^{-1}r)\exp(-\Omega_0^{-1}(r^2+b_N^2))\,I_0(2\Omega_0^{-1}b_N r), \qquad (2)$$

where $I_0(.)$ is the zero-th order modified Bessel function. Let $U=R^2$ and $V_N=B_N^2$, which upon substitution into (2) results in the following unconditional PDF for U

$$f_U(u)=E_{V_N}[\Omega_0^{-1}\exp(-\Omega_0^{-1}(u+V_N))\,I_0(2\Omega_0^{-1}\sqrt{V_N u})], \qquad (3)$$

where E is expectation. The BER expression of several modulations in additive white Gaussian noise is an exponential function of U [5]. Here we consider binary differential phase shift keying whose BER, conditioned on U, is given by $P(U)=0.5\exp(-\gamma_b U)$, where γ_b is the signal to noise ratio (SNR) per bit. Using (3), average BER with N specular components, $\overline{P}_N=\int_0^\infty P(u)\,f_U(u)du$, can be written as

$$\begin{aligned}\overline{P}_N&=E_{V_N}\left[\frac{1}{2\Omega_0}\int_0^\infty\exp\left(-\gamma_b u-\frac{u+V_N}{\Omega_0}\right)I_0\left(\frac{2}{\Omega_0}\sqrt{V_N u}\right)du\right]\\&=E_{V_N}\left[\frac{1}{2(\gamma_b\Omega_0+1)}\exp\left(\frac{-\gamma_b V_N}{\gamma_b\Omega_0+1}\right)\right],\end{aligned} \qquad (4)$$

where the integral in (4) is analytically solved using Mathematica.

A. Lowest possible BER

Because $\gamma_b(\gamma_b\Omega_0+1)^{-1}>0$, the term $\exp(-\gamma_b(\gamma_b\Omega_0+1)^{-1}V_N)$ in (4) is a convex function in V_N. Therefore, Jensen's inequality [9] results in

$$\overline{P}_N\geq 0.5(\gamma_b\Omega_0+1)^{-1}\exp\left(-\gamma_b(\gamma_b\Omega_0+1)^{-1}\Omega_N\right), \qquad (5)$$

where V_N in the exponential in (4) is replaced by $E[V_N]=\Omega_N=\sum_{i=1}^{N}a_i^2$, to obtain the right-hand side of the inequality in (5). The lower bound in (5) holds for any N, and can be achieved for $N=1$. This is because according to eq. (8.213) in [5], BER in Rice fading is exactly the same as the right-hand side of (5). This completes our proof that for a fixed total power of specular components Ω_N, Rice fading results in the lowest possible BER.

B. Recursive formula for BER

Without loss of generality and to simplify the notation, let $\Phi_1=0$. For $N=2$, the sum of two vectors (specular components) results in $B_2\exp(j\Psi_2)=a_1+a_2\exp(j\Phi_2)$. Using the cosine formula in the triangle with sides B_2, a_1 and a_2, we obtain $B_2^2=a_1^2+a_2^2+2a_1a_2\cos\Phi_2$. With $V_2=B_2^2$ and $V_1=a_1^2$, this can be written as $V_2=V_1+a_2^2+2a_2\sqrt{V_1}\cos\Phi_2$. When the third vector $a_3\exp(j\Phi_3)$ is added, we can similarly write $V_3=V_2+a_3^2+2a_3\sqrt{V_2}\cos(\Phi_3-\Psi_2)$, conditioned on V_2 and Ψ_2. By adding vectors one by one we obtain the recursion $V_i=V_{i-1}+a_i^2+2a_i\sqrt{V_{i-1}}\cos(\Phi_i-\Psi_{i-1})$, $i=2,3,\ldots,N$, with $\Psi_1=0$ and conditioned on V_{i-1} .and Ψ_{i-1}. This approach is used in [10] to derive the amplitude pdf recursively, whereas we use it in a different way, to obtain new BER results, by computing the BER recursively. By substituting $V_N=V_{N-1}+a_N^2+2a_N\sqrt{V_{N-1}}\cos(\Phi_N-\Psi_{N-1})$ into (4) we obtain the following BER, conditioned on V_{N-1}, Ψ_{N-1} and Φ_N, as shown in (6).

$$\begin{aligned}P_N(V_{N-1},\Psi_{N-1},\Phi_N)&=\frac{1}{2(\gamma_b\Omega_0+1)}\\&\times\exp\left(-\frac{\gamma_b(V_{N-1}+a_N^2+2a_N\sqrt{V_{N-1}}\cos(\Phi_N-\Psi_{N-1}))}{\gamma_b\Omega_0+1}\right).\end{aligned} \qquad (6)$$

Based on the integral identity $I_0(x)=(2\pi)^{-1}\int_0^{2\pi}\exp(x\cos(\beta-\xi))d\beta$, eq. (6) can be averaged with respect to Φ_N, which is uniformly distributed over $[0,2\pi)$. Since the resulting expression depends only on V_{N-1}, the recursive average BER can be ultimately written as (7), where $V_0\equiv 0$. Eq. (7) includes Rice BER, eq. (8.213) in [5] as a special case, for $N=1$. When $N=2$, since $V_1=a_1^2$ is a constant, eq. (7) results in (8).

$$\overline{P}_N=\frac{1}{2(\gamma_b\Omega_0+1)}E_{V_{N-1}}\left[\begin{array}{l}\exp\left(\dfrac{-\gamma_b}{\gamma_b\Omega_0+1}(V_{N-1}+a_N^2)\right)\\ \times I_0\left(\dfrac{2\gamma_b a_N}{\gamma_b\Omega_0+1}\sqrt{V_{N-1}}\right)\end{array}\right],\quad N=1,2,3,\ldots \qquad (7)$$

$$\bar{P}_2 = 0.5(\gamma_b\Omega_0+1)^{-1}\exp\left(-\gamma_b(\gamma_b\Omega_0+1)^{-1}(a_1^2+a_2^2)\right) \times I_0\left(2\gamma_b(\gamma_b\Omega_0+1)^{-1}a_1a_2\right). \quad (8)$$

For $N=3$ the expectation in (7) is over V_2, which depends on Φ_2. Therefore

$$\bar{P}_3 = 0.5(\gamma_b\Omega_0+1)^{-1}E_{\Phi_2}\left[\exp\left(-\gamma_b(\gamma_b\Omega_0+1)^{-1}(V_2+a_3^2)\right) \times I_0\left(2\gamma_b(\gamma_b\Omega_0+1)^{-1}a_3\sqrt{V_2}\right)\right]. \quad (9)$$

Following the same approach, eq. (7) for $N=4$ ultimately results in

$$\bar{P}_4 = 0.5(\gamma_b\Omega_0+1)^{-1}E_{\Phi_3,\Phi_2}\left[\exp\left(-\gamma_b(\gamma_b\Omega_0+1)^{-1}(V_3+a_4^2)\right) \times I_0\left(2\gamma_b(\gamma_b\Omega_0+1)^{-1}a_4\sqrt{V_3}\right)\right], \quad (10)$$

with V_3 defined before and $\sqrt{V_2}\cos(\Phi_3-\Psi_2) = (a_1+a_2\cos\Phi_2)\cos\Phi_3 + a_2\sin\Phi_2\sin\Phi_3$. Equations (8)-(10) are computationally more efficient than integrals over products of Bessel functions (see [8] and references therein).

C. *Highest and lowest BERs using Lagrange multipliers*

The BER in (8) is a function of a_1 and a_2. For a fixed total specular power $a_1^2+a_2^2=\Omega_2$, BER varies as a_1 and a_2 change, and it is of interest to find out what values of a_1 and a_2 maximize or minimize the BER. This a constrained optimization problem, subject to the constraint $a_1^2+a_2^2=\Omega_2$. To solve this using Lagrange multipliers [11], we define the Lagrange function $L(a_1,a_2,\alpha)=\bar{P}_2+\alpha(a_1^2+a_2^2-\Omega_2)$, where α is the Lagrange multiplier. By setting partial derivatives of L zero, we obtained three critical points. They agree with the points in [12], derived using a BER expression that is entirely different from the new recursive BER in (7). The BER expression in [12] is not efficient for numerical computations. Also it is not analytically investigated there which point gives the maximum BER and which one gives the minimum.

The points $(a_1,a_2)=(\sqrt{\Omega_2},0)$ and $(0,\sqrt{\Omega_2})$ indicate that the specular power is in one component only, which is Rice fading and is shown in (5) to be the lowest BER. The third critical point $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$ indicates two equal amplitude specular components. Using the Hessian matrix [10], we have shown the BER is maximum at this critical point. Detailed derivations are provided in Appendix.

III. Numerical Results

Let $K=\Omega_N/\Omega_0$ be the specular to diffuse power ratio and $\mu_i=a_i^2/a_1^2$, $i=2,3,...,N$ be the i-th specular power ratio. Also consider unit total power, i.e., $\Omega_N+\Omega_0=1$, without loss of generality. Using eq. (8), Fig. 1 shows the BER for $N=2$ versus μ_2 with specular power ratio $K=5\ dB$, 15 dB and 20 dB. In agreement with the theoretical derivation, BER is maximum when the two specular components have equal strength, that is, $\mu_2=1$, and is minimum when one specular component is much stronger than the other, which means $\mu_2=0$ or ∞.

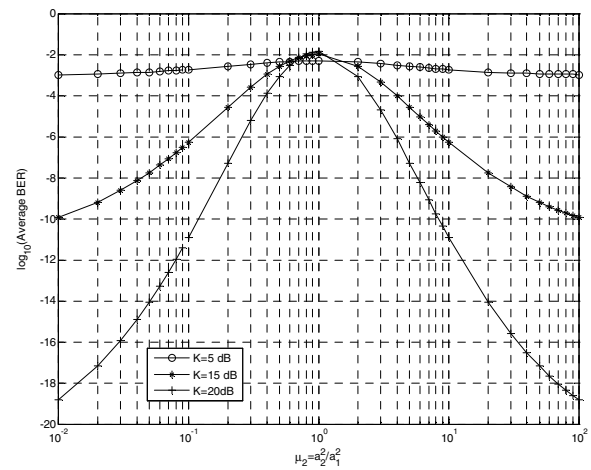

Fig. 1. Bit error rate in a fading channel with two specular components versus the specular power ratio, with $\gamma_b=20$ dB and different specular to diffuse power ratios K.

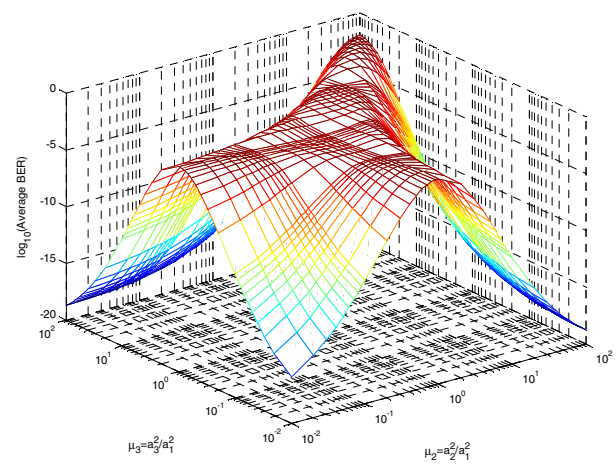

Fig. 2. Bit error rate in a fading channel with three specular components versus the specular power ratios, with $\gamma_b=20$ dB and specular to diffuse power ratio $K=20$ dB.

The BER for $N=3$ is plotted in Fig. 2 versus μ_2 and μ_3 with $K=20\ dB$ and $\gamma_b=20\ dB$, using eq. (9). Consistent with the analytical finding that minimum BER occurs when one specular component is much stronger than the others, we observe that the minima in Fig. 2 are located at $(\mu_2,\mu_3)=(0.01,0.01),(100,0.01),(0.01,100)$. The maxima appear

to occur when $(\mu_2, \mu_3) = (100,100),(1,0.01),(0.01,1)$. This agrees with the constrained optimization result that maximum BER for two specular components happens when the amplitudes are equal.

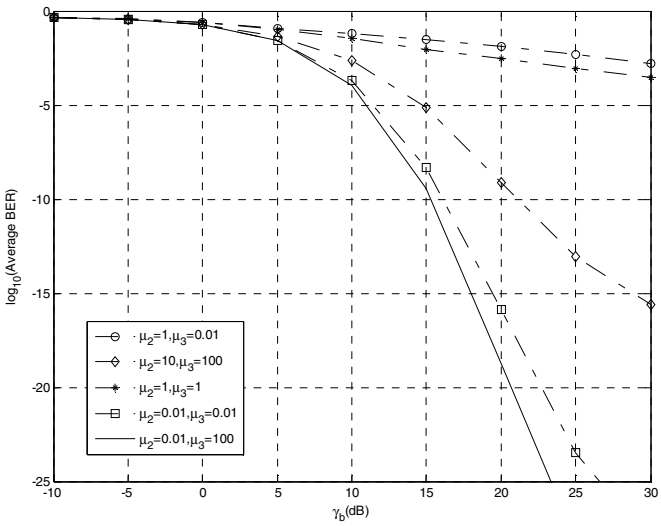

Fig. 3. Bit error rate in a fading channel with three specular components versus SNR, with different specular power ratios μ_2 and μ_3 and the specular to diffuse power ratio $K = 20$ dB.

More numerical results are shown in Fig. 3 and 4 with different specular power ratios μ_i for $N = 3, 4$. Two strong specular components provide higher BERs and one strong specular component gives lower BERs. To study the impact of *N*, in Fig. 5 BER is plotted for equal-strength specular components, using (8)-(10). Unit power Rayleigh fading BER, $0.5(\gamma_b + 1)^{-1}$ [5], is also plotted as a reference. For a fixed total specular power, as shown in (5) using Jensen's inequality, $N = 1$ (Rice fading) has the lowest BER. Moreover, $N = 2$ has the highest BER. Interestingly, BER for $N = 4$ is higher than the BER for $N = 3$ at high SNRs. As *N* increases, BER approaches the Rayleigh case.

IV. Conclusion

In this paper a new BER formula is derived for multipath channels with $N \geq 1$ specular paths. Upon using this formula, it is proved that $N = 1$, Rice fading, provides the lowest BER among all possible values for *N*. For equal-amplitude specular components, the numerical results show that with a fixed total specular power, BERs lie between $N = 1$ and $N = 2$ curves. Moreover, $N = 4$ provides a BER higher than $N = 3$, specially at high SNRs.

These results are useful for system design and performance prediction in underwater acoustic channels containing a direct ray as well as few surface and bottom reflected rays in early arrivals. However, the findings and derived formulas are not limited to underwater acoustic channels and systems. Another case where the results are of interest is a spatially selective system with directive antennas, where only a limited number of waves from certain directions might be received [6]. Similar situation can occur in wideband systems where a short delay bin may contain only a small number of multipath components [13]. The results could also be of interest in vehicular wireless sensor networks [14].

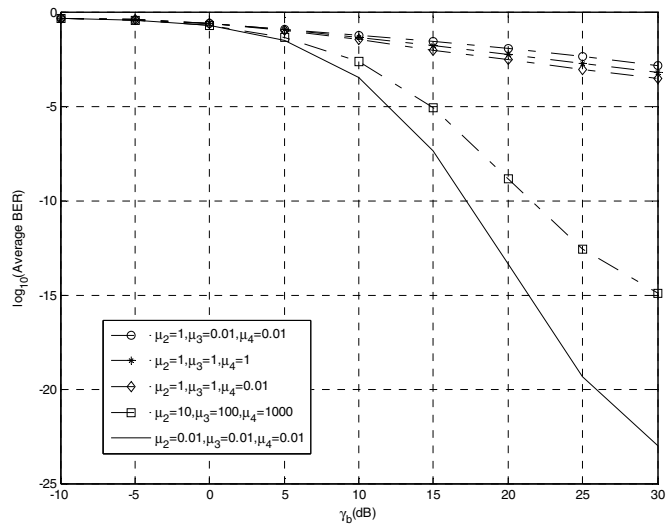

Fig. 4. Bit error rate in a fading channel with four specular components versus SNR, with different specular power ratios μ_2 , μ_3 and μ_4 and the specular to diffuse power ratio $K = 20$ dB.

Appendix
The BER Constrained Optimization Problem and Lagrange Multipliers

With two specular components, BER in multipath fading channels is given by (8). Define the Lagrange function as

$$L(a_1, a_2, \alpha) = 0.5(\gamma_b \Omega_0 + 1)^{-1} e^{\frac{-\gamma_b}{\gamma_b \Omega_0 + 1}\Omega_2} I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b \Omega_0 + 1}\right) + \alpha(a_1^2 + a_2^2 - \Omega_2). \quad (11)$$

Taking the partial derivatives of L with respect to a_1 and a_2 , respectively, results in

$$\partial L(a_1, a_2, \alpha)/\partial a_1 = e^{\frac{-\gamma_b}{\gamma_b \Omega_0 + 1}\Omega_2} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b \Omega_0 + 1}\right)\frac{\gamma_b a_2}{(\Omega_0 \gamma_b + 1)^2} - e^{\frac{-\gamma_b}{\gamma_b \Omega_0 + 1}\Omega_2} I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b \Omega_0 + 1}\right)\frac{\gamma_b a_1}{(\Omega_0 \gamma_b + 1)^2} + 2\alpha a_1 = 0,$$

$$\partial L(a_1, a_2, \alpha)/\partial a_2 = e^{\frac{-\gamma_b}{\gamma_b \Omega_0 + 1}\Omega_2} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b \Omega_0 + 1}\right)\frac{\gamma_b a_1}{(\Omega_0 \gamma_b + 1)^2}$$

$$-e^{\frac{-\gamma_b}{\gamma_b\Omega_0+1}\Omega_2} I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b a_2}{(\Omega_0\gamma_b+1)^2}+2\alpha a_2=0. \quad (12)$$

Solving the two equations above provides

$$e^{\frac{-\gamma_b}{\gamma_b\Omega_0+1}\Omega_2} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b(a_1^2-a_2^2)}{(\gamma_b\Omega_0+1)^2}=0, \quad (13)$$

which leads to three critical points. they are: $(a_1,a_2)=(\sqrt{\Omega_2},0)$, $(0,\sqrt{\Omega_2})$ and $(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$. At the points $(a_1,a_2)=(\sqrt{\Omega_2},0)$ and $(0,\sqrt{\Omega_2})$, the specular power is allocated to a single component only, which is Rice fading . As shown in (5), BERs at these two points are minimum.

To determine whether the point $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$ gives a minimum or a maximum BER, we need to solve an equation based on the Hessian matrix [11], where the equation is given by (14).

The determinant on the right hand side of (14) needs to be computed at $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$. From (12), we know at the point $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$ we have (15).

$$\begin{vmatrix} \partial^2 L(a_1,a_2,\alpha)/\partial a_1^2 - x & \partial^2 L(a_1,a_2,\alpha)/\partial a_1\partial a_2 & 2\alpha a_1 \\ \partial^2 L(a_1,a_2,\alpha)/\partial a_2\partial a_1 & \partial^2 L(a_1,a_2,\alpha)/\partial a_2^2 - x & 2\alpha a_2 \\ 2\alpha a_1 & 2\alpha a_2 & 0 \end{vmatrix} = 0. \quad (14)$$

$$\alpha=-0.5e^{\frac{-\gamma_b}{\gamma_b\Omega_0+1}\Omega_2} I_1\left(\frac{\gamma_b\Omega_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b}{(\Omega_0\gamma_b+1)^2} + 0.5e^{\frac{-\gamma_b}{\gamma_b\Omega_0+1}\Omega_2} I_0\left(\frac{\gamma_b\Omega_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b}{(\Omega_0\gamma_b+1)^2}. \quad (15)$$

Moreover, second derivatives of L are given in (16) and (17). Upon substitution of (15)-(17) into (14), the roots of (14) can be shown to be $x_1=x_2=-2\gamma_b\exp(-\gamma_b\Omega_2/(\gamma_b\Omega_0+1)) \times I_1\left(\gamma_b\Omega_2/(\gamma_b\Omega_0+1)\right)/(\Omega_0\gamma_b+1)^2<0$. The negative sign [11] indicates the point $(a_1,a_2)=(\sqrt{\Omega_2/2},\sqrt{\Omega_2/2})$ corresponds to the maximum BER. This agrees with the numerical results in Fig. 1.

$$\partial^2 L(a_1,a_2,\alpha)/\partial a_1\partial a_2 = \partial^2 L(a_1,a_2,\alpha)/\partial a_2\partial a_1 = 0.5e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}}\left(3I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)+I_2\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\right)\frac{(\gamma_b)^2\Omega_2}{(\gamma_b\Omega_0+1)^3} -2e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{(\gamma_b)^2\Omega_2}{(\gamma_b\Omega_0+1)^3} +e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b}{(\gamma_b\Omega_0+1)^2}, \quad (16)$$

$$\partial^2 L(a_1,a_2,\alpha)/\partial a_1^2 = \partial^2 L(a_1,a_2,\alpha)/\partial a_2^2 = 0.5e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}}\left(3I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)+I_2\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\right)\frac{(\gamma_b)^2\Omega_2}{(\gamma_b\Omega_0+1)^3} -2e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}} I_1\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{(\gamma_b)^2\Omega_2}{(\gamma_b\Omega_0+1)^3} -e^{\frac{-\gamma_b\Omega_2}{\gamma_b\Omega_0+1}} I_0\left(\frac{2\gamma_b a_1 a_2}{\gamma_b\Omega_0+1}\right)\frac{\gamma_b}{(\Omega_0\gamma_b+1)^2}+2\alpha. \quad (17)$$

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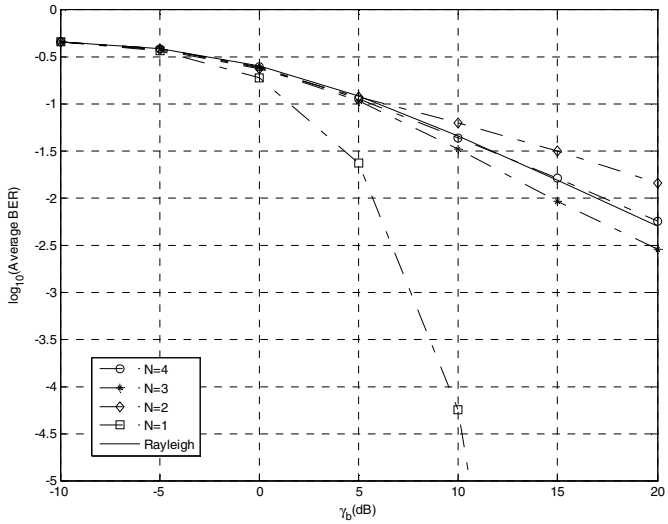

Fig. 5. Bit error rate in a fading channel with $N = 1, 2, 3, 4$ specular components versus SNR, with equal amplitudes, $K = 20$ dB, and the corresponding Rayleigh bit error rate.