PERFORMANCE ANALYSIS OF MOMENT-BASED ESTIMATORS FOR THE K PARAMETER OF THE RICE FADING DISTRIBUTION

Cihan Tepedelenlioğlu, Ali Abdi, Georgios B. Giannakis, and Mostafa Kaveh

Dept. of Elec. and Comp. Engr., Univ. of Minnesota, Minneapolis, MN 55455, U.S.A.

ABSTRACT

In mobile communications the strength of a line of sight component measured by the K factor of the Ricean received envelope distribution has significant impact on system performance analysis and link budget calculations. In this paper, we study the performance of moment-based estimators for the Ricean K-factor as less complex alternatives to the maximum likelihood estimator. Our asymptotic analysis reveals that the estimators that rely on lower-order moments have a better asymptotic performance for moderate/large values of K. We also illustrate, by Monte Carlo simulations, that the fading correlation among the envelope samples deteriorates the estimator performance. The simplest estimator, which can be expressed in closed form in terms of the second- and fourth-order sample moments offers a good compromise between statistical performance and computational simplicity.

1. INTRODUCTION

In mobile communications, when a line of sight (LOS) component is present between the transmitter and the receiver, the received signal is given by the sum of a sinusoid and narrow-band Gaussian noise, whose envelope is known to have the Ricean distribution (see e.g., [9]). The Rice probability density function (PDF) of the received envelope $R(t)$ is given by:

$$p_R(r) = \frac{2(K+1)r}{\Omega}\exp\left(-K-\frac{(K+1)r^2}{\Omega}\right) \times I_0\left(2r\sqrt{\frac{K(K+1)}{\Omega}}\right), \tag{1}$$

where $I_n(\cdot)$ is the n^{th} order modified Bessel function of the first kind, $K \geq 0$ is the Ricean factor and $\Omega := E[R^2(t)]$. The K factor is given by the ratio of the LOS component's power to the power of the narrow-band Gaussian noise, and its estimation is important in link budget calculations [3], and for determination of the channel quality. Simple estimation techniques for the K-factor are also of importance in the optimization of transmit diversity schemes and adaptive transmission systems (see, e.g. [4] for the estimation of the Nakagami m parameter, which is a similar measure of channel quality, for transmitter diversity optimization). Notice that when $K = 0$ there is no LOS component, in which case the distribution of the received envelope reduces to Rayleigh.

The maximum-likelihood estimator (MLE) for the K parameter from independent and identically distributed (i.i.d.) samples of the envelope entails first calculating the MLE estimator for Ω, which is given by $\hat{\Omega}_{ML} = \hat{\mu}_2$, where $\hat{\mu}_n := N^{-1}\sum_{k=0}^{N-1} R^n(kT_s)$ denotes the n^{th} sample moment, T_s is the sampling period, and N is the number of available samples [1]. The MLE for K can then be obtained by substituting $\hat{\Omega}_{ML}$ for Ω in the likelihood function and maximizing the resulting nonlinear equation with respect to K [10]. However, such a solution is computationally complex. The expectation maximization algorithm has been proposed in [6] to reduce the complexity, but it is still not easy to use. The distribution-fitting approaches in [3] provide robust, but nevertheless, computationally complex alternatives, which are not easy to implement online. In this paper, we will investigate the performance of moment-based estimators for K which are simpler than the alternatives mentioned above.

2. MOMENT-BASED ESTIMATORS FOR K

The moments of the Ricean distribution are given by [9]:

$$\mu_n := E[R^n(t)] = (2\sigma^2)^{n/2}\,\Gamma\left(n/2+1\right)\,\exp(-K) \times {}_1F_1\left(n/2+1;1;K\right), \tag{2}$$

where $2\sigma^2$ is the power of the narrow-band noise and can be shown to be equal to $\Omega/(K+1)$, ${}_1F_1(\cdot;\cdot;\cdot)$ is the confluent hypergeometric function, and $\Gamma(\cdot)$ is the gamma function. We see from (2) that the moments depend on the two unknown parameters K and σ. Hence, a moment-based estimator will require estimates of two different moments of $R(t)$. More specifically, suppose that for $n \neq m$ we define the following functions of K (recall that μ_n is the n^{th}

This work is supported in part by the National Science Foundation, under the Wireless Initiative Program, Grant No. 9970443.

[1] We will drop the dependence of $\hat{\mu}_n$ on N for notational convenience.

moment of $R(t)$):

$$f_{n,m}(K) := \frac{\mu_n^m}{\mu_m^n} \ . \tag{3}$$

Since $f_{n,m}(K)$ depends only on K and not on σ we can construct moment-based estimators for K by using sample moments instead of the ensemble values in (3) and then inverting the corresponding $f_{n,m}(K)$, to solve for K. Hence, an estimator that depends on the m^{th} and n^{th} moments could be expressed as:

$$\hat{K}_{n,m} := f_{n,m}^{-1}\left(\frac{\hat{\mu}_n^m}{\hat{\mu}_m^n}\right) \ , \tag{4}$$

provided that the inverse function $f_{n,m}^{-1}(\cdot)$ exists. For all the values of m and n we considered, $f_{n,m}(K)$ is a monotone increasing function in the interval $K \in (0, \infty)$, and hence the inverse function $f_{n,m}^{-1}(\cdot)$ exists.

The natural choice for (n, m) is $(1, 2)$ since this selection involves the lowest order moments. When $n = 1$ and $m = 2$, (3) can be calculated using (2) as:

$$f_{1,2}(K) = \frac{\pi e^{-K}}{4(K+1)}[(K+1)I_0(\frac{K}{2}) + KI_1(\frac{K}{2})]^2 \ . \tag{5}$$

The corresponding estimator $\hat{K}_{1,2}$ involves the complex numerical procedure of inverting (5). This estimator has been discussed in [7] and its performance was studied in detail in [10] via simulations where it was found that $\hat{K}_{1,2}$ performs similarly to the MLE.

A simpler alternative to $\hat{K}_{1,2}$ is $\hat{K}_{2,4}$. It can be shown using (2) and (3) that

$$f_{2,4}(K) = \frac{(K+1)^2}{K^2 + 4K + 2} \ . \tag{6}$$

Clearly, calculating $f_{2,4}^{-1}(K)$ involves finding the roots of a second-order polynomial which can be done in closed form. It can be shown that one of the roots of this polynomial is always negative which can be discarded since $K > 0$, yielding a unique positive solution for $\hat{K}_{2,4}$ which is given by:

$$\hat{K}_{2,4} = \frac{-2\hat{\mu}_2^2 + \hat{\mu}_4 + \hat{\mu}_2\sqrt{2\hat{\mu}_2^2 - \hat{\mu}_4}}{\hat{\mu}_2^2 - \hat{\mu}_4} \ . \tag{7}$$

The estimator in (7) has been independently proposed in [2] and [8], though not presented in this form. In what follows, we will derive asymptotic variance (AsV) expressions for $\hat{K}_{n,m}$ specifically focusing on the performance of $\hat{K}_{1,2}$ as compared with $\hat{K}_{2,4}$.

3. ASYMPTOTIC VARIANCE OF K-ESTIMATORS

Using the results in [5, pp. 60] it can be shown that for the moment-based estimators $\hat{K}_{n,m}$, $\lim_{N\to\infty}\sqrt{N}(\hat{K}_{n,m} - K)$ is a Gaussian random variable with mean zero, and variance given by:

$$\begin{aligned}
\lim_{N\to\infty} \quad N \quad \mathrm{var}\left(\hat{K}_{n,m} - K\right) & \\
= \quad & \left(\frac{m\mu_m^{-n}\mu_n^{m-1}}{f'_{n,m}(K)}\right)^2 (\mu_{2n} - \mu_n^2) \\
- \quad & 2\frac{m\mu_m^{-n}\mu_n^{m-1}}{f'_{n,m}(K)}\frac{n\mu_m^{-n-1}\mu_n^m}{f'_{n,m}(K)}(\mu_{n+m} - \mu_n\mu_m) \\
+ \quad & \left(\frac{n\mu_m^{-n-1}\mu_n^m}{f'_{n,m}(K)}\right)^2 (\mu_{2m} - \mu_m^2) \ ,
\end{aligned} \tag{8}$$

where $f'_{n,m}(K)$ is the derivative of $f_{n,m}(K)$ with respect to K, and can be computed using (2) and (3). All moment-based estimators, $\hat{K}_{n,m}$, are $\sqrt{N}$-consistent, asymptotically normal, and asymptotically unbiased [1]. Moreover, the AsV is given by (8). Notice that $\hat{K}_{n,m} = \hat{K}_{m,n}$ which can be shown using the fact that $f_{n,m}^{-1}(x) = f_{m,n}^{-1}(1/x)$. Interchanging m and n in (8), then, should not change the AsV, which is seen to be the case after computing (8) fully as a function of K.

In order to compare the AsV expression in (8) with a benchmark, we numerically computed the Cramer-Rao bound (CRB), which provides a lower bound for the variance of any unbiased estimator. The CRB was also reported in [10], and is given by:

$$\mathrm{CRB}(K) = \frac{1}{\int_0^\infty \left(\frac{\partial}{\partial K}[\ln p_R(r)]\right)^2 p_R(r)dr} \ . \tag{9}$$

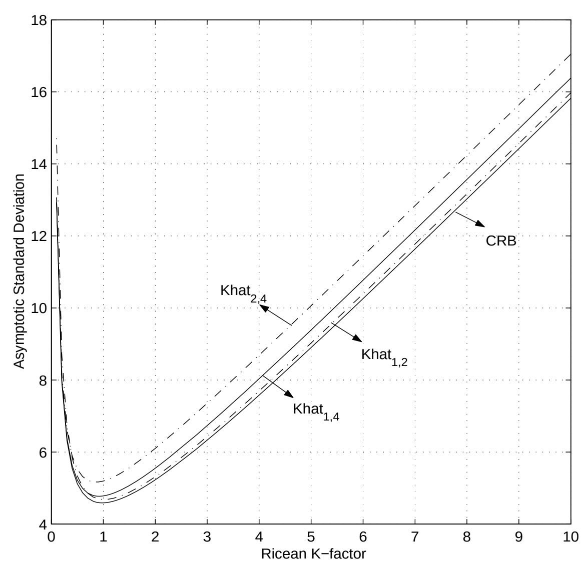

Fig. 1. Asymptotic Performance of K-estimators

In Figure 1, we plotted the asymptotic standard deviation (std) expressions for $\hat{K}_{1,2}$, $\hat{K}_{1,4}$, and $\hat{K}_{2,4}$ as well as the square root of the CRB to understand the performance of moment-based estimators for large sample sizes. We notice that as K gets smaller and smaller, making the Ricean pdf more and more like Rayleigh, no estimator can estimate K accurately, because the CRB goes to infinity. We also observe that the most accurate estimation of K is possible around $K = 1$, and as K increases, the square-root of the AsV goes to infinity approximately linearly. Among the moment-based estimators, $\hat{K}_{1,2}$ has the least AsV for moderate/large K, and is in fact very close to the CRB, which leads us to conclude that $\hat{K}_{1,2}$ is *almost* asymptotically efficient. As we expected, increasing m and n result in larger AsV for moderate/large K. Indeed, the simple estimator $\hat{K}_{2,4}$ in (7), for which there is a closed form expression, has a greater AsV than $\hat{K}_{1,4}$ or $\hat{K}_{1,2}$ have. Another interesting observation is that $\hat{K}_{1,4}$ has smaller AsV than $\hat{K}_{1,2}$ for small values of $K < 1$, even though it employs higher-order moments. The difference between the asymptotic variances of $\hat{K}_{1,4}$ and $\hat{K}_{1,2}$ for $K < 1$ is too small to be visible in Figure 1. In order to see this difference, we plotted the asymptotic stds of $\hat{K}_{1,2}$, $\hat{K}_{1,3}$, $\hat{K}_{1,4}$, and $\hat{K}_{2,4}$ minus the square root of the CRB in Figure 2 for small values of K. We observe that the estimators which employ higher-order moments are better asymptotically, for sufficiently small K. The practical significance of this result is that, if we have large data samples available, and we would like to detect the presence or absence of a LOS component (in which case the value of K would typically be small), we might be interested in employing $\hat{K}_{2,4}$ instead of $\hat{K}_{1,2}$, to perform the hypothesis testing.

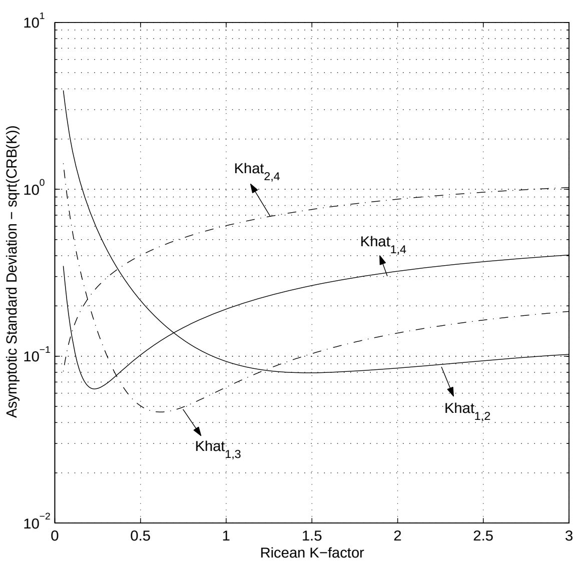

Fig. 2. Asymptotic Performance of K-estimators (small K)

4. EFFECT OF FINITE SAMPLE SIZE

In order to study the effect of finite sample size on the performance of $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$, we resorted to Monte Carlo simulations. For any fixed K from the set $\{0.5, 1, 1.5, \ldots, 19.5, 20\}$, broad enough to cover a practical range of the K parameter [2], and for any $N \in \{100, 1000\}$, 500 sequences of i.i.d. samples of length N were generated for $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$. Let $\hat{K}(j)$ denote the j^{th} Monte Carlo realization of either $\hat{K}_{1,2}$ or $\hat{K}_{2,4}$. For both of these estimators the sample bias $500^{-1}\sum_{j=1}^{500}[\hat{K}(j) - K]$ is plotted in Figure 3 versus K together with the sample confidence region defined by ± 2 SSTD($\hat{K}$), where SSTD($\hat{K}$) is the sample std of $\hat{K}$, defined as:

$$\mathrm{SSTD}(\hat{K}) := \sqrt{500^{-1}\sum_{j=1}^{500}\hat{K}^2(j) - (500^{-1}\sum_{j=1}^{500}\hat{K}(j))^2} \; .$$

The sample confidence region defined here is useful for examining the estimator variations in terms of K and N.

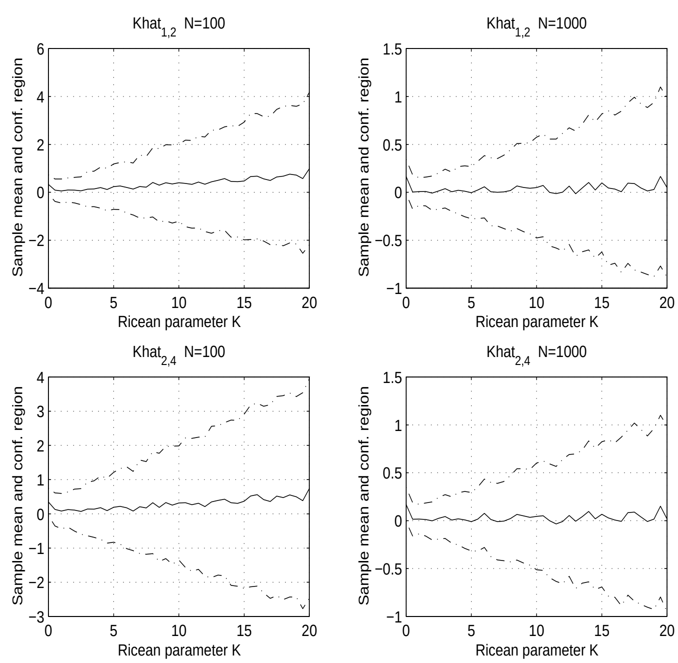

Fig. 3. Monte Carlo simulation for finite sample sizes

In Figure 3, we observe that, as expected, the confidence region and the sample bias is smaller for larger sample sizes. We also observe that for moderate/large K, increasing K increases the bias especially for $N = 100$ for both estimators. Finally, we point out that the performance of $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$ are very similar for both sample sizes.

5. THE EFFECT OF CORRELATION

In practice, adjacent signal samples can be highly correlated. To analyze the impact of correlated samples on the performance of $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$, we again used Monte Carlo simulations. Using the same simulation procedure as before and for $N = 1,000$, we generated 500 Rice distributed envelope time-series whose corresponding in-phase and quadrature components have the Clarke-Jakes's correlation function [9], $J_0(2\pi f_D \tau)$, where $J_0(\cdot)$ is the zeroth order Bessel function of the first kind, and f_D is the maximum Doppler frequency. Figure 4 shows the simulation results for two different mobile speeds (different f_Ds), at a sampling rate of $1/T_s = 243$ Hz corresponding to samples taken from an IS-136 system every 100 symbols. For both estimators, the correlation among samples, which increases with decreasing mobile speed, introduces a positive bias which grows with K and also broadens the sample confidence region (more estimator variation). Based on the simulation results, we conclude that $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$ still perform similarly even for correlated samples, and that the samples should be chosen far apart to avoid the deleterious effects of correlation on the estimates.

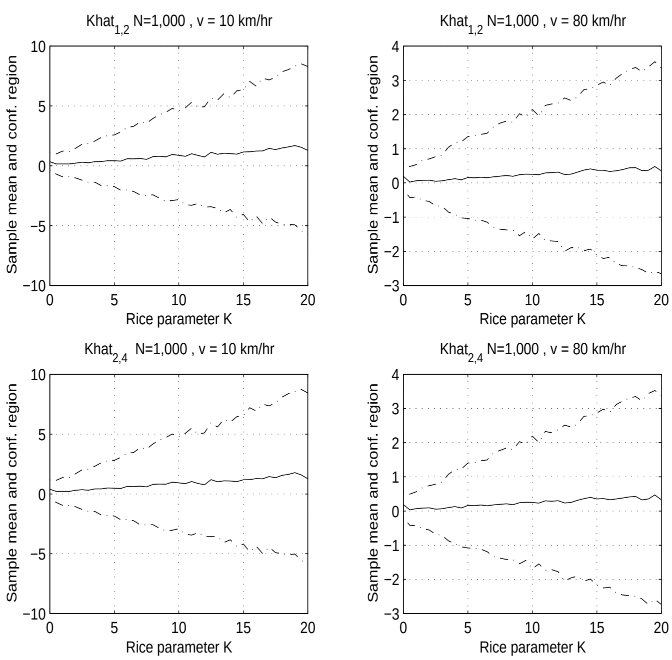

Fig. 4. Monte Carlo simulation for correlated samples

6. CONCLUSIONS

In this paper, we have studied the performance of moment-based estimators for the Ricean K-factor, that are simple alternatives to the MLE. We have derived an expression for the asymptotic variance (AsV) given in (8). Specifically we were interested in $\hat{K}_{1,2}$ because it involves the lowest order sample moments, and $\hat{K}_{2,4}$ because it has a closed-form expression in terms of the sample moments. We have seen that the AsV of $\hat{K}_{1,2}$ is very close to the CRB and smaller than the AsV of $\hat{K}_{2,4}$ for moderate/large K. We have also observed that for small values of K, moment-based estimators that rely on higher-order moments can have smaller variance.

The effect of finite sample size is also investigated via Monte Carlo simulations, which led us to conclude that $\hat{K}_{1,2}$ and $\hat{K}_{2,4}$ perform similarly also when the sample size is small. The study of correlation effects suggests that for low mobile speeds (small Doppler spread), which introduces significant correlation among the envelope samples, closely-spaced samples deteriorate the estimators' performance. In a nutshell, $\hat{K}_{2,4}$ in (7) offers a good compromise between computational convenience and statistical efficiency and could be recommended for practical implementation.

7. REFERENCES

[1] H. Cramer, *Mathematical Methods of Statistics*, Princeton, NJ: Princeton University Press, 1946.

[2] L. J. Greenstein, D. G. Michelson, and V. Erceg, "Moment-method estimation of the Ricean K-factor," *IEEE Communication Letters*, vol. 3, pp. 175-176, 1999.

[3] D. Greenwood and L. Hanzo, "Characterization of mobile radio channels," in *Mobile Radio Communications*. R. Steele, Ed., London: Pentech, 1992, pp. 92-185.

[4] Y. C. Ko and M. S. Alouini, "Estimation of Nakagami-m fading channel parameters with application to optimized transmitter diversity systems", *International Conference on Communications*, Helsinki, Finland, 2001 (submitted).

[5] E. L. Lehmann and G. Casella, *Theory of Point Estimation*, 2nd ed., New York: Springer, 1998.

[6] T. L. Marzetta, "EM algorithm for estimating the parameters of a multivariate complex Ricean density for polarimetric SAR," in *Proc. ICASSP*, Detroit, MI, 1995, pp. 3651-3654.

[7] R. Prasad, *Universal Wireless Personal Communications*, Artech House, 1998.

[8] P. K. Rastogi and O. Holt, "On detecting reflections in presence of scattering from amplitude statistics with application to *D* region partial reflections," *Radio Science*, vol. 3, pp. 175-176, 1999.

[9] G. L. Stuber, *Principles of Mobile Communication*, Boston, MA: Kluwer, 1996.

[10] K. K. Talukdar and W. D. Lawing, "Estimation of the parameters of the Rice distribution," *Journal of Acoustical Society of America*, vol. 89, pp. 1193-1197, 1991.