Statistical Characterization of Eigen-Channels in Time-Varying Rayleigh Flat Fading MIMO Systems

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Abstract—In this paper, some important second-order statistics such as the correlation coefficient, level crossing rate, and average fade duration of eigen-channels, are studied, in time-varying Rayleigh multiple-input multiple-output (MIMO) channels, assuming a general non-isotropic scattering environment. Exact closed-form expressions are derived and Monte Carlo simulations are provided to verify the accuracy of our closed-form results. These results serve as useful tools for analysis and design of MIMO systems in time-varying channels.

I. INTRODUCTION

The utilization of antenna arrays at the base station (BS) and the mobile station (MS) in a wireless communication system increases the capacity linearly with min(N_T, N_R), where N_T and N_R are numbers of transmit and receive antenna elements, respectively, provided that the environment is sufficiently rich in multi-path components [1][2]. This is due to the fact that a multiple-input multiple-output (MIMO) channel can be decomposed to several parallel single-input single-output (SISO) channels, called eigen-channels or eigen-modes, via the singular value decomposition (SVD) [2]–[9].

For a SISO channel, or any subchannel of a MIMO system, there are numerous studies on key second-order statistics such as correlation, level crossing rate (LCR), and average fade duration (AFD) [10]–[14]. However, to the best of our knowledge regarding the joint probability density function (PDF) of eigen-channels, these statistics will provide useful metrics for designing the adaptive transmission scheme over eigen-channels.

In this paper, a number of second-order statistics such as the autocorrelation function (ACF), correlation coefficient, LCR and AFD of eigen-channels are studied in MIMO time-varying Rayleigh flat fading channels. We assume all the subchannels are spatially independent and identically distributed (i.i.d.), with the same temporal correlation coefficient, considering general non-isotropic scattering propagation environments. Closed-form expressions are derived, and Monte Carlo simulations are provided to verify the accuracy of our closed-form expressions.

Notation: $^T$ is reserved for matrix Hermitian, $^*$ for complex conjugate, $j$ for $\sqrt{-1}$, $E[\cdot]$ for mathematical expectation, $I_m$ for the $m \times m$ identity matrix, $\| \cdot \|$ for the Frobenius norm, $\Re[\cdot]$ and $\Im[\cdot]$ for the real and imaginary parts of a complex number, respectively, and $f^2(x)$ for $|f(x)|^2$. Finally, $t \in [m, n]$ implies that $t$, $m$ and $n$ are integers such that $m \leq t \leq n$ with $m \leq n$.

The rest of this paper is organized as follows. Sec. II introduces the channel model, as well as the angle-of-arrival (AoA) model. Eigen-channels of a MIMO system are discussed in Sec. III. Sec. IV is devoted to the derivation of the normalized ACF (NACF) and correlation coefficient the eigen-channels of a $2 \times 2$ MIMO system, whereas Sec. V focuses on LCR and AFD of the eigen-channels. Numerical results are presented in Sec. VI, and concluding remarks are given in Sec. VII.

II. CHANNEL MODEL

In this paper, an $N_R \times N_T$ MIMO time-varying Rayleigh flat fading channel is considered. Similar to [15], we consider a piecewise constant approximation for the continuous-time MIMO fading channel matrix coefficient $H(t)$, represented by \{H(I_T)\}_{t=1}^{L}$, where $T$ is the symbol duration and $L$ is the number of samples. In the $l$th symbol duration, the matrix of channel coefficients is given by

$$H(I_T) = \begin{bmatrix}
h_{1,1}(I_T) & \cdots & h_{1,N_T}(I_T) \\
\vdots & \ddots & \vdots \\
h_{N_R,1}(I_T) & \cdots & h_{N_R,N_T}(I_T)
\end{bmatrix}, l \in [1, L]. \quad (1)$$

We assume all the $N_T N_R$ subchannels $\{h_{n_r,n_t}(I_T), l \in [1, L]\}_{n_r=1}^{N_R}, n_t=1$ are i.i.d., with the same temporal correlation coefficient, i.e.,

$$\mathbb{E}[h_{m,n}(I_T)h_{m,n}^*(l-i)T_s] = \delta_{m,p}\delta_{n,q}\rho_h(iT_s), \quad (2)$$

where the Kronecker delta $\delta_{m,p}$ is 1 or 0 when $m = p$ or $m \neq p$, respectively, and $\rho_h(iT_s)$ is derived at the end of this section, eq. (6).

In flat Rayleigh fading channels, each $h_{n_r,n_t}(I_T), l \in [1, L]$, is a zero-mean complex Gaussian random process. In the $l$th interval, $h_{n_r,n_t}(I_T)$ can be represented as [13]

$$h_{n_r,n_t}(I_T) = h_{n_r,n_t}^T(I_T) + jh_{n_r,n_t}^Q(I_T),$$

$$= R_{n_r,n_t}(I_T) \exp[-j\Phi_{n_r,n_t}(I_T)], \quad (3)$$
where the zero-mean real Gaussian random processes \( h_{n_r,n_s}(lT_b) \) and \( h_{n_r,n_s}^Q(lT_b) \) are the real and imaginary parts of \( h_{n_r,n_s}(lT_b) \), respectively. \( \alpha_{n_r,n_s}(lT_b) \) is the envelope of \( h_{n_r,n_s}(lT_b) \) and \( \Phi_{n_r,n_s}(lT_b) \) is the phase of \( h_{n_r,n_s}(lT_b) \). For each \( l, \alpha_{n_r,n_s}(lT_b) \) has a Rayleigh distribution and \( \Phi_{n_r,n_s}(lT_b) \) is distributed uniformly over \([0, \pi)\). Without loss of generality, we assume each subchannel has unit power, i.e., \( \mathbb{E} \left[ \alpha_{n_r,n_s}^2(lT_b) \right] = 1 \).

Using the empirically-verified [13] multiple von Mises PDF [16, (4)] for the AoA at the receiver in non-isotropic scattering environments, shown as Fig. 1 of [16], the channel correlation coefficient of \( h_{n_r,n_s}(lT_b), \forall n_r,n_s \), is given by [16, (7)]

\[
\rho_h(iT_s) = \mathbb{E}[h_{n_r,n_s}(lT_b)h_{n_r,n_s}^*(l-iT_s)],
\]

\[
= \sum_{n=1}^{N} P_n \sqrt{\kappa_n^2 - 4\pi^2 f_b^2 T_s^2} \text{J}_0(\kappa_n^2) \mathcal{I}_0(\kappa_n),
\]

where \( \text{J}_0(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(k \theta) d\theta \) is the \( k \)th order modified Bessel function of the first kind, \( \theta_n \) is the mean AoA of the \( n \)th cluster of scatterers, \( \kappa_n \) controls the width of the \( n \)th cluster of scatterers, \( P_n \) represents the contribution of the \( n \)th cluster of scatterers such that \( \sum_{n=1}^{N} P_n = 1 \), \( 0 < P_n \leq 1 \), \( N \) is the number of clusters of scatterers, and \( f_b \) is the maximum Doppler frequency. When \( \kappa_n = 0 \), \( \forall n \), which corresponds to isotropic scattering, (6) reduces to \( \rho_h(iT_s) = \text{J}_0(2\pi f_b T_s) = \text{J}_0(2\pi f_b T_s) \) [16], which is the Clarke’s correlation model.

### III. Eigen-Channels in MIMO Systems

We set \( M = \min(N_T, N_R) \) and \( N = \max(N_T, N_R) \). Based on singular value decomposition (SVD) [2][9], \( \mathbf{H}(lT_b) \) in (1) can be diagonalized in the following form

\[
\mathbf{H}(lT_b) = \mathbf{U}(lT_b) \mathbf{S}(lT_b) \mathbf{V}^H(lT_b),
\]

where \( \mathbf{V}(lT_b) \), whose dimension is \( N_T \times M \), satisfies \( \mathbf{V}^H(lT_b) \mathbf{V}(lT_b) = \mathbf{I}_M \), which is \( N_R \times M \), satisfies \( \mathbf{U}^H(lT_b) \mathbf{U}(lT_b) = \mathbf{I}_M \), and \( \mathbf{S}(lT_b) \) is a diagonal matrix, given by \( \mathbf{S}(lT_b) = \text{diag}[s_1(lT_b), \ldots, s_M(lT_b)] \), in which \( s_n(lT_b), m \in [1, M] \) is the \( m \)th non-zero singular value of \( \mathbf{H}(lT_b) \).

We define \( \lambda_m(lT_b) = s_m^2(lT_b), \forall m \). Therefore \( \lambda_m(lT_b) \) is the \( m \)th non-zero eigenvalue of \( \mathbf{H}(lT_b) \mathbf{V}^H(lT_b) \). We further consider \( \left\{ \lambda_m(lT_b) \right\}_{m=1}^{M} \) as unordered non-zero eigenvalues of \( \mathbf{H}(lT_b) \mathbf{H}^H(lT_b) \). Therefore the MIMO channel \( \mathbf{H}(lT_b) \) is decomposed into \( M \) identically distributed\(^2\) eigen-channels, \( \left\{ \lambda_m(lT_b), l \in [1, L] \right\}_{m=1}^{M} \), by SVD, as shown in Fig. 1. For \( M = 1 \), there is only one eigen-channel, which corresponds to the maximum ratio transmitter (MRT) if \( N_R = 1 \), or the maximum ratio combiner (MRC) if \( N_T = 1 \). In each case we have \( N \) i.i.d complex Gaussian branches.

Since all the eigen-channels have identical statistics, we only study one of them and denote it as \( \lambda(lT_b), l \in [1, L] \). To simplify the notation, we use \( X \) and \( Y \) to denote \( \lambda(lT_b) \) and \( \lambda(l-iT_b) \), respectively. The joint PDF of \( X \) and \( Y \) is given in (4) [17],

\[
L_n^0(x) = \text{J}_0(2\pi \text{f}_0 \text{T}_b) = \text{J}_0(2\pi \text{f}_0 \text{T}_b) \quad \text{[16]},
\]

which is the associated Laguerre polynomial of order \( n \) [18, pp. 1061, 8.9701.1], \( \nu = N - M \), and \( \text{f}_0 = \left| \rho_h(lT_b) \right| \), where \( \rho_h(lT_b) \) is given in (6). The joint PDF in (4) is very general and includes many existing PDF’s as special cases.

- **By integration over \( y \), (4) reduces to the marginal PDF**

\[
p(x) = \frac{1}{M} \sum_{m=0}^{M-1} \frac{m!}{(m+\nu)!} \left[ L_m^0(x) \right]^2 x^\nu e^{-x},
\]

which is the same as the PDF presented in [2]. When \( M = 1 \), (8) further reduces to

\[
p(x) = \frac{1}{(N-1)!} x^{N-1} e^{-x},
\]

which is the \( \chi^2 \) distribution with \( 2N \) degrees of freedom [19, (2.32)], used for characterizing the PDF of outputs of MRT or MRC.

- **With \( M = 1 \), (4) reduces to**

\[
p(x,y) = \frac{(xy)^{N-1} \exp \left( -\frac{x+y}{1-\text{g}_i^2} \right) \text{I}_{N-1} \left( \frac{2\sqrt{x+y}}{1-\text{g}_i^2} \right)}{(N-1)! (1-\text{g}_i^2)^{N-1}},
\]

\(^{2}\)In general, the eigen-channels are correlated.
which is the joint PDF of outputs of MRT or MRC at the $l$th and $(l-i)$th symbol durations [20]. Furthermore, when $N = 1$, i.e., a SISO channel, (10) simplifies to
\begin{equation}
p(x, y) = \frac{1}{1 - g_i^2} \exp\left(-\frac{x+y}{1 - g_i^2}\right) I_0\left(\frac{2\eta_i \sqrt{xy}}{1 - g_i^2}\right),
\end{equation}
which is identical to (8-103) [21, pp. 163] after a one-to-one nonlinear mapping.

In the following sections, the normalized autocorrelation, correlation coefficient, LCR and AFD of an eigen-channel $\lambda(T_x), l \in [1, L]$ are investigated for a $2 \times 2$ MIMO system\(^3\), where the joint PDF is given in (5), after plugging $M = 2$ and $\nu = 0$ into (4).

IV. NACF AND CORRELATION COEFFICIENT OF AN EIGEN-CHANNEL

In this section, first we concentrate on the ACF of an eigen-channel $\lambda(T_x), l \in [1, L]$. The ACF is defined by
\begin{equation}
r_{\lambda}(i) = \mathbb{E}[\lambda(T_x)\lambda((l-i)T_x)],
\end{equation}
where $p(x, y)$ is given by (5).

Combining (5) with the following Taylor series [18, pp. 971, 8.447.1]
\begin{equation}
I_0(t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(k!)^2 2^{2k}},
\end{equation}
we simplify (13) to
\begin{equation}
r_{\lambda}(i) = \begin{cases} 8, & i = 0, \\ 4 + g_i^2, & i \neq 0, \end{cases}
\end{equation}
where the second moment of $X$, $r_{\lambda}(0)$, is calculated as $\mathbb{E}[X^2]$ directly, using the PDF of $X$, which can be derived from (8) by replacing $M$ with 2
\begin{equation}
p(x) = \left(\frac{x^2}{2} - x + 1\right) e^{-x}, x \geq 0.
\end{equation}

From (15), we can easily get NACF as
\begin{equation}
\tilde{r}_{\lambda}(i) = \frac{r_{\lambda}(i)}{\mathbb{E}[X^2]} = \begin{cases} 1, & i = 0, \\ \frac{g_i^2}{4}, & i \neq 0. \end{cases}
\end{equation}

With $\mathbb{E}[X] = 2$, the mean of $X$, which is easily calculated using (16), the correlation coefficient is given by
\begin{equation}
\rho_{\lambda}(i) = \frac{r_{\lambda}(i) - \mathbb{E}[X^2]}{\sqrt{\mathbb{E}[X^2] - \mathbb{E}[X]^2}} = \begin{cases} 1, & i = 0, \\ \frac{g_i^2}{4}, & i \neq 0. \end{cases}
\end{equation}

From (17) and (18), it is interesting to observe that NACF and the correlation coefficient are not continuous at $i = 0$, as $\tilde{r}_{\lambda}(1)$ and $\rho_{\lambda}(1)$ do not converge to $\tilde{r}_{\lambda}(0) = \rho_{\lambda}(0) = 1$ as $T_x \to 0$.

\(^3\)Other configurations can be handled via the same techniques developed in this paper, and using the joint PDF in (4). The $2 \times 2$ case is emphasized in this paper to provide further insight, rather than lengthy equations.

With isotropic scattering, (17) and (18), respectively, reduce to
\begin{equation}
\tilde{r}_{\lambda}(i) = \begin{cases} 1, & i = 0, \\ \frac{1}{2} + \frac{\sqrt{\rho}}{2}, & i \neq 0, \end{cases}
\end{equation}
and
\begin{equation}
\rho_{\lambda}(i) = \begin{cases} 1, & i = 0, \\ \frac{\sqrt{\rho}}{2}, & i \neq 0. \end{cases}
\end{equation}

V. LCR AND AFD OF AN EIGEN-CHANNEL

A. LCR of an Eigen-Channel

Similar to the calculation of zero crossing rate in discrete time [22, Ch. 4], we define the binary sequence $\{Z_l\}_{l=1}^L$, based on the eigen-channel samples $\lambda(T_x)_{l=1}^L$, as
\begin{equation}
Z_l = \begin{cases} 1, & \text{if } \lambda(T_x) \geq \lambda_h, \\ 0, & \text{if } \lambda(T_x) < \lambda_h, \end{cases}
\end{equation}
where $\lambda_h$ is a fixed threshold. The number of crossings of $\lambda(T_x)_{l=1}^L$ with $\lambda_h$, within the time interval $T_s \leq t \leq LT_s$, denoted by $D_{\lambda_h}$, can be defined in terms of $\{Z_l\}_{l=1}^L$ [22, (4.1)]
\begin{equation}
D_{\lambda_h} = \sum_{l=2}^{L} (Z_l - Z_{l-1})^2,
\end{equation}
which includes both up- and down-crossings.

After some simple manipulations, the expected crossing rate at the level $\lambda_h$ can be written as
\begin{equation}
\mathbb{E}[D_{\lambda_h}] = \frac{2P_r\{Z_l = 1\} - 2P_r\{Z_l = 1, Z_{l-1} = 1\}}{T_s}.
\end{equation}

To simplify the notation, we use $\phi$ for $P_r\{Z_l = 1\}$ and $\varphi(\phi)$ to denote $P_r\{Z_l = 1, Z_{l-1} = 1\}$, where $\varphi(\phi) = |\rho_{\lambda}(T_x)|$, defined before, i.e.,
\begin{equation}
\phi = \sum_{n=1}^{N} P_r(\sqrt{\kappa_n^2 - 4\pi f_B^2 T_s^2 + 4\pi \kappa_n f_B T_s \cos \theta_n}) I_{0}(\kappa_n),
\end{equation}
and
\begin{equation}
\varphi(\phi) = P_r\{X \geq \lambda_h, Y \geq \lambda_h\},
\end{equation}
which simplifies to (12), with $\Gamma(a, z) = \int_{z}^{\infty} t^{a-1} e^{-t} dt$ [18, pp. 949, 8.350.2] as the upper incomplete gamma function, by applying (5) and (14) to (27).
\[
\varphi(q_1) = \frac{(1-q_1^2)^3}{4q_1^2} \sum_{k=0}^{\infty} \frac{q_1^{2k}}{(1-q_1^2)^2} \left\{ \frac{q_1^2}{(1-q_1^2)^2} \Gamma^2(k+1, \lambda_{th} \frac{1}{q_1}) + \frac{1}{1-q_1^2} \Gamma(k+1, \lambda_{th} \frac{1}{q_1}) - \Gamma(k+2, \lambda_{th} \frac{1}{q_1}) \right\} \\
+ e^{-2\lambda_{th}} \left[ \frac{1}{2} - \frac{\lambda_{th}^2}{4} \left( \frac{1}{q_1} - \frac{1}{\varphi(q_1)} \right)^2 \right]
\]

By plugging (26) and (27) into (24), we obtain the expected crossing rate at the level \( \lambda_{th} \) as
\[
N_\lambda(\lambda_{th}) = \frac{e^{-\lambda_{th}} \left( 1 + \frac{\lambda_{th}^2}{2} \right) - \varphi(q_1)}{T_s}, \tag{28}
\]
where \( \varphi(q_1) \) is given in (12).

**B. AFD of an Eigen-Channel**

The cumulative distribution function (CDF) of \( \lambda(lT_s) \), \( \forall l \) is obtained as
\[
F_\chi(\lambda_{th}) = P_r \{ X \leq \lambda_{th} \} = 1 - \varphi = 1 - e^{-\lambda_{th}} \left( 1 + \frac{\lambda_{th}^2}{2} \right), \tag{29}
\]
where \( \varphi \) comes from (26).

The AFD of an eigen-channel is therefore given by
\[
\overline{t}_\lambda(\lambda_{th}) = \frac{F_\chi(\lambda_{th})}{N_\lambda(\lambda_{th})} = \frac{1 - e^{-\lambda_{th}} \left( 1 + \frac{\lambda_{th}^2}{2} \right)}{e^{-\lambda_{th}} \left( 1 + \frac{\lambda_{th}^2}{2} \right) - \varphi(q_1)}, \tag{30}
\]
where \( \varphi(q_1) \) is given in (12).

**VI. NUMERICAL RESULTS AND DISCUSSION**

In this paper, a generic power spectrum [16, (8)] is used to simulate Rayleigh flat fading channels with non-isotropic scattering, according to the spectral method [23]. To verify the accuracy of the derived formulas, we consider two types of scattering environments: isotropic scattering, and non-isotropic scattering with three clusters of scatterers. For the non-isotropic scattering, parameters of three clusters, \([P_n, \kappa_n, \theta_n], n = 1, 2, 3\), are given by \([P_1, \kappa_1, \theta_1] = [0.45, 2, \frac{\pi}{3}], [P_2, \kappa_2, \theta_2] = [0.2, 20, \frac{11\pi}{18}]\), and \([P_3, \kappa_3, \theta_3] = [0.35, 3, \frac{13\pi}{36}]\), respectively. In addition, in all the simulations, the maximum Doppler frequency \( f_D \) is set to 1Hz, and \( T_s = \frac{1}{v_0} \) seconds.

In the following subsections, simulations are performed to verify NACF, correlation coefficient, LCR and AFD of an eigen-channel of a \( 2 \times 2 \) MIMO system, in the above two propagation environments.

**A. Isotropic Scattering**

This is Clarke’s model [10], with uniform AoA. The simulation results are shown in Fig. 2.

\footnote{Note that the value of \( f_D \) is has just a scaling effect, \( f_D = 1 \) Hz is chosen to make the simulations faster.}

**B. Non-isotropic Scattering**

This is a general case, with an arbitrary AoA distribution. Simulations are carried out, with the results shown in Fig. 3.

In Figs. 2-3, the upper left and right figures show NACF and correlation coefficient of an eigen-channel, in a \( 2 \times 2 \) MIMO system with isotropic scattering.

**Fig. 2.** The channel correlation coefficient, AoA distribution, as well as NACF and correlation coefficient of an eigen-channel, in a \( 2 \times 2 \) MIMO system with isotropic scattering.

**Fig. 3.** The channel correlation coefficient, AoA distribution, as well as NACF and correlation coefficient of an eigen-channel, in a \( 2 \times 2 \) MIMO system with non-isotropic scattering.
The amplitude of the channel correlation coefficient evaluated at \( \tau = T_a \).

From Figs. 2-5, one can say that for both types of scattering, the theoretical formulas perfectly match the simulation results.

VII. CONCLUSION

In this paper, closed-form expressions for the autocorrelation function, correlation coefficient, level crossing rate and average fade duration of an eigen-channel are derived, for a 2 \( \times \) 2 MIMO time-varying Rayleigh flat fading channel. The analytical expressions, supported by Monte Carlo simulations, provide useful qualitative and quantitative information regarding the fluctuations of MIMO channels.

Although we only considered 2 \( \times \) 2 MIMO systems, it is straightforward to use the techniques developed in this paper to deal with other MIMO systems with \( N_T > 2 \) transmitters and \( N_R > 2 \) receivers, where all the subchannels are independent and identically distributed, with the same temporal correlation function.

REFERENCES


