AN OPTIMUM LAGUERRE EXPANSION FOR THE ENVELOPE PDF OF TWO SINE WAVES IN GAUSSIAN NOISE

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Abstract - The sum of two randomly-phased sine waves and Gaussian noise arises in various fields of communications. In this paper, a Laguerre series and also a power series are introduced, for the envelope PDF of this random process. Moreover, tight upper bounds are derived for the truncation error of these two infinite series. Comparison of these two upper bounds show that the Laguerre series is superior to the power series; because for a fixed number of terms, it yields minimum truncation error.

INTRODUCTION

In many situations of communication engineering, the signal of interest consists of the sum of two randomly-phased sine waves, with uniform distributions, and Gaussian noise. A single tone perturbed by a jammer tone [1], signals received through media with a double propagation path (*i.e.* propagation over sea water [2], or by double ionospheric layers [3]), etc. can be considered as typical examples. Thus, calculation of receiver performance for such cases, depends upon knowledge of the statistical properties of two randomly-phased sine waves, with uniform distributions, in Gaussian noise. In this paper, first order probability density function (PDF) for the envelope of this important stochastic process is investigated in detail.

CLOSED-FORM FORMULAS FOR $f_R(r)$

It can be shown that the envelope PDF, $f_R(r)$, has the following form [4]:

$$f_R(r) = r\int_0^\infty \lambda J_0(r\lambda)\exp(-\frac{\lambda^2}{2})J_0(a_1\lambda)J_0(a_2\lambda)d\lambda \quad (1)$$

where a_1 and a_2 are the amplitudes of the sine waves, and R is the one-dimensional random variable of the envelope process, all normalized to the standard deviation of the noise. In the above formula, $J_0(.)$ is the Bessel function of order zero.

The sum of two sine waves with amplitudes a_1 and a_2 and uniformly distributed phases, results in a single sine wave with amplitude A and uniformly distributed phase. Note that in contrast to a_1 and a_2, which are deterministic constants, A is a random variable, with the following PDF [4]:

$$f_A(a) = 2a[\pi\sqrt{(a^2 - a_{\min}{}^2)(a_{\max}{}^2 - a^2)}\,]^{-1};\ a \in]a_{\min}, a_{\max}[\quad (2)$$

in which:

$$a_{\min} = a_1 - a_2 \geq 0 \quad a_{\max} = a_1 + a_2 \quad (3)$$

Based on the above reasoning, the problem of two sine waves in Gaussian noise reduces to the problem of a single sine wave in Gaussian noise. Thus, conditioned on A, R becomes a Rice random variable, with the following conditional PDF [5]:

$$f_{R|A}(r|a) = r\exp(-\frac{r^2 + a^2}{2})I_0(ar) \quad (4)$$

where $I_0(.)$ is the modified Bessel function of order zero. In this way, $f_R(r)$ in (1) can also be expressed in another form:

$$f_R(r) = r\exp(-\frac{r^2}{2})E_A[\exp(-\frac{A^2}{2})I_0(Ar)] \quad (5)$$

At the end of this section, it is necessary to introduce the following Laguerre generating function [6]:

$$\exp(\sigma)J_0(2\sqrt{\tau\sigma}) = \sum_{n=0}^\infty \frac{L_n(\tau)\sigma^n}{n!} \quad (6)$$

which will be used in the subsequent sections. In (6), $L_n(.)$ is the Laguerre polynomial of order n [6]. In addition, defining $h_n(z)$ as:

$$h_n(z) = E_A[\exp(-zA^2)A^{2n}] \quad (7)$$

will simplify subsequent formulas, considerably. Using (2), it can be shown that [7]:

$$h_n(z) = \frac{1}{\pi}\exp(-a_{\min}{}^2 z)a_{\min}{}^{2n}$$

$$\times \sum_{j=0}^n \frac{n!}{(n-j)!\,j!}(\frac{4a_1a_2}{a_{\min}{}^2})^j B(\frac{1}{2}, j+\frac{1}{2})$$

$$\times {}_1F_1(j+\frac{1}{2}, j+1, -4a_1a_2z) \quad (8)$$

where B denotes the beta function, and ${}_1F_1$ stands for the confluent hypergeometric function [6].

LAGUERRE SERIES FOR $f_R(r)$

For $\tau = -\beta r^2/4$ and $\sigma = A^2/\beta$, and noting that $I_0(z) = J_0(iz)$, (6) gives the following parametric Laguerre series for $I_0(Ar)$, in terms of *r*:

$$I_0(Ar) = \exp(-\frac{A^2}{\beta})\sum_{n=0}^{\infty}\frac{1}{n!}L_n(-\frac{\beta r^2}{4})(\frac{A^2}{\beta})^n;\ \beta \neq 0 \qquad (9)$$

By inserting (9) into (5), one obtains the following parametric Laguerre series for $f_R(r)$:

$$f_R(r) = \sum_{n=0}^{\infty} w_n(\beta) g_n(\beta, r);\ \beta \neq 0 \qquad (10)$$

in which:

$$w_n(\beta) = \frac{1}{n!\beta^n} E_A[A^{2n}\exp(-(\frac{1}{2}+\frac{1}{\beta})A^2)] \qquad (11)$$

$$g_n(\beta, r) = r\exp(-\frac{r^2}{2})L_n(-\frac{\beta r^2}{4}) \qquad (12)$$

Based on (7), $w_n(\beta)$ can also be expressed as:

$$w_n(\beta) = \frac{1}{n!\beta^n} h_n(\frac{1}{2}+\frac{1}{\beta}) \qquad (13)$$

It should be mentioned that (10) is general in the sense that it has the arbitrary parameter β. In fact, for $\beta = -2$ and -4, (10) reduces to [4, (12)] and [8, (3)], respectively.

POWER SERIES FOR $f_R(r)$

For $\tau = A^2/\beta$ and $\sigma = -\beta r^2/4$, and noting that $I_0(z) = J_0(iz)$, (6) gives the following parametric power series for $I_0(Ar)$, in terms of *r*:

$$I_0(Ar) = \exp(\frac{\beta r^2}{4})\sum_{n=0}^{\infty}\frac{1}{n!}L_n(\frac{A^2}{\beta})(-\frac{\beta r^2}{4})^n; \beta \neq \infty \qquad (14)$$

By inserting (14) into (5), one obtains the following parametric power series for $f_R(r)$:

$$f_R(r) = \sum_{n=0}^{\infty} v_n(\beta) d_n(\beta, r);\ \beta \neq \infty \qquad (15)$$

in which:

$$v_n(\beta) = \frac{1}{n!} E_A[\exp(-\frac{A^2}{2})L_n(\frac{A^2}{\beta})] \qquad (16)$$

$$d_n(\beta, r) = r\exp(-(\frac{1}{2}-\frac{\beta}{4})r^2)(-\frac{\beta r^2}{4})^n \qquad (17)$$

Based on (7), and using the definition of $L_n(.)$ [6]:

$$L_n(z) = \sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!(k!)^2}z^k \qquad (18)$$

$v_n(\beta)$ can also be expressed as:

$$v_n(\beta) = \sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!(k!)^2\beta^k}h_k(\frac{1}{2}) \qquad (19)$$

It should be mentioned that (15) is general in the sense that it has the arbitrary parameter β. However, in contrast to (10), it has not been studied previously.

DERIVING UPPER BOUNDS FOR THE TRUNCATION ERROR OF (10) AND (15)

It is clear that in practice, just a finite number of terms, say $n_{\max}$, can be considered in an infinite series. In this way, a truncation error is introduced. Usually, it is of interest to select $n_{\max}$ in such a way to maintain the absolute value of the truncation error below a specified value. To fullfill this requirement, it is necessary to obtain a relationship between $n_{\max}$ and the absolute value of the truncation error. In what follows, such relationships are presented for (10), the Laguerre series, and (15), the power series. However, instead of the truncation errors, their upper bounds are considered, in order to obtain mathematically tractable results.

Truncation Error Upper Bound of Laguerre Series [7]

Consider $q_n(\beta)$ and $u_n(\beta)$ as:

$$q_n(\beta) \geq |w_n(\beta)| \qquad u_n(\beta) \geq \max_r |g_n(\beta, r)| \qquad (20)$$

Now, upper bound for the absolute value of the truncation error in (10) can be expressed as:

$$e_L = \sum_{n=n_{\max}+1}^{\infty} q_n(\beta) u_n(\beta) \qquad (21)$$

In the above formula, one has:

$$q_n(\beta) = \frac{1}{n!\sqrt{\pi}}\exp(4a_1a_2\left|\frac{1}{2}+\frac{1}{\beta}\right| - a_{\min}{}^2(\frac{1}{2}+\frac{1}{\beta}))(\frac{a_{\max}{}^2}{|\beta|})^n \qquad (22)$$

$$u_n(\beta) = \frac{1}{n!}\sqrt{(1-\frac{2}{\beta})2n}\exp((\frac{2}{\beta}-1)n)(-\frac{\beta}{2}n)^n;\ \beta \leq -4$$

$$= \frac{1}{\sqrt{e}};\ -4 < \beta < 0$$

$$= \sqrt{\frac{2\beta n}{\beta+2}}\exp(-\frac{\beta n}{\beta+2})L_n(-\frac{\beta^2 n}{2(\beta+2)});\ \beta > 0 \qquad (23)$$

It should be mentioned that in contrast to (22), (23) holds only for medium and large values of *n*, i.e. $n \geq 20$.

Truncation Error Upper Bound of Power Series **[7]**

Consider $t_n(\beta)$ and $s_n(\beta)$ as:

$$t_n(\beta) \geq |v_n(\beta)| \quad s_n(\beta) \geq \max_r |d_n(\beta, r)| \tag{24}$$

Now, upper bound for the absolute value of the truncation error in (15) can be expressed as:

$$e_P = \sum_{n=n_{\max}+1}^{\infty} t_n(\beta) s_n(\beta) \tag{25}$$

In the above formula, one has:

$$t_n(\beta) = \frac{1}{n!\sqrt{\pi}} \exp(3a_1 a_2 - \frac{a_1^2}{2} - \frac{a_2^2}{2}) L_n(-\frac{a_{\max}^2}{|\beta|}) \tag{26}$$

$$s_n(\beta) = \frac{2}{\sqrt{|\beta|}} (\frac{|\beta|}{2-\beta} \frac{2n+1}{2e})^{n+1/2}; \ \beta < 2 \tag{27}$$

It should be mentioned that (26) and (27) hold for every *n*.

OPTIMUM β IN (10) AND (15)

It can be shown that except for $\beta \neq 0$ and $\beta \neq \infty$, (10) and (15) converge for every β [7]. So this natural question arises that for each series, which value of β is optimum. In [7], and for a fixed $n_{\max}$, maximizing the convergence rate of (21) and (25) is considered as the optimization goal. In fact, it is shown that $\beta = -3.9$ and $\beta = 0$ maximize the convergence rate of (21) and (25), respectively.

COMPARISON OF TWO OPTIMUM SERIES

It is shown in [7] that:

$$\lim_{n\to\infty} \frac{q_{n+1}(-3.9) u_{n+1}(-3.9)}{q_n(-3.9) u_n(-3.9)} = \frac{a_{\max}^2}{3.9} \lim_{n\to\infty} \frac{1}{n} = 0 \tag{28}$$

$$\lim_{n\to\infty} \frac{t_{n+1}(0) s_{n+1}(0)}{t_n(0) s_n(0)} = \frac{a_{\max}^2}{2} \lim_{n\to\infty} \frac{1}{n} = 0 \tag{29}$$

Clearly, the convergence rate of (28) is approximately twice. From this point of view, the optimum Laguerre series is more efficient than the optimum power series. Thus, for calculating the envelope PDF of two sine waves in Gaussian noise, it is better to use (10) with $\beta = -3.9$.

It should be mentioned that as a function of *n*, $q_n(-3.9)$ exhibits a unimodal behavior. The location of its peak, n_p, is given by:

$$n_p = 0.26 a_{\max}^2 \tag{30}$$

For large values of a_1 and a_2, the peak of $q_n(-3.9)$ becomes very large. So, to obtain a small e_L, $n_{\max}$ in (21) should be sufficiently greater than n_p.

CONCLUSION

In this paper, formula (10) and (15) are introduced for the envelope PDF of two randomly-phased sine waves in Gaussian noise. Formula (21) and (25) are tight upper bounds for their truncation errors. As the main result, it is shown that (10), the Laguerre series, in superior to (15), the power series.

The Laguerre series has been used previously for the above envelope PDF, but in a suboptimal manner ([4], [8]). However, the Laguerre series of this paper is optimal, in the sense that for a predetermined truncation error, it requires minimum value of $n_{\max}$. As an example, consider that $a_1 = 14$, $a_2 = 10$, and it is desired to have $e_L \approx 2E-15$. For $\beta = -3.9$, the required $n_{\max}$ is 540; while $\beta = -2$ and -4 yield $n_{\max} = 810$ and 619, respectively.

It should be mentioned that similar results are presented in [9], for a more general case in which *N* sine waves with correlated random amplitudes are combined with noise. However, its results are not applicable for $N = 2$, the case studied completely in this paper.

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