A GENERAL PDF FOR THE SIGNAL ENVELOPE IN MULTIPATH FADING CHANNELS USING LAGUERRE POLYNOMIALS

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Abstract- Multipath fading usually degrades the performance of communication systems, severely. To describe this random phenomenon, it is necessary to derive expressions for the PDF of the received signal envelope in multipath fading channels. In this paper, two expansions for the envelope PDF's are derived under general channel characteristics: an infinite power series and an infinite Laguerre series. Uniform upper bounds are derived for the truncation errors of these two infinite series. Moreover, these uniform upper bounds are minimized using a free parameter β, defined consciously. It has also been shown that the Laguerre series is superior to the power series, in the sense that for a fixed number of terms, it yields less truncation error. Based on an extensive literature survey, reported partly in the paper, it seems that such closed form expressions for the envelope PDF has not been derived previously, under the general conditions considered here.

I. INTRODUCTION

In a variety of situations encountered in the field of communication engineering, the structure of the transmission channel is such that the received signal exhibits strong fluctuations in its amplitude. These fluctuations are generally referred to as fading phenomena and a great deal of attention has been devoted to the study of their properties and to devising receiving schemes capable of reducing their undesired effects. Whenever a satisfactory characterization of the fading phenomena (and other channel related issues) can be achieved, the problem of optimum and suboptimum reception in the presence of noise can be formulated in the frame of detection theory.

When a single unmodulated carrier (i.e. with constant envelope) is transmitted in a multipath fading channel, it breaks into several multipath components. These multipath components add vectorially (according to their amplitudes and phases), and a rapidly fluctuating envelope is experienced by the receiver. Generally speaking, and for a fixed instant of time, the number of multipath components, the amplitudes, and the phases of multipath components are all random variables. Thus, due to the multipath fading phenomenon, the constant envelope of the transmitted carrier changes to a stochastic process; and so, a probabilistic model should be developed to describe this rapidly fluctuating envelope. A useful probabilistic model is the random vector model, since each multipath component can be considered as a random vector, with random length and angle.

If a closed form formula for the envelope probability density function (PDF) in a multipath fading channel is available, then it can be used to predict analytically the performance of communication systems with various modulation schemes, in the presence of multipath fading. It also facilitates the design of accurate channel simulators. Thus, it is very important to obtain a mathematically tractable formula for $f_R(r)$, the envelope PDF, under general conditions.

In this paper, two equivalent envelope PDF's are derived based on the following assumptions: deterministic number of multipath components, and iid phases (Φ_i's), with uniform distributions. Note that amplitudes (A_i's) can be arbitrary positive dependent random variables. These two equivalent PDF's are compared thoroughly based on their numerical properties, and the best one is introduced.

For the case of random number of multipath components, see [1]-[8]; and for non-uniform phase distributions, see [9] and references therein.

II. THE MODEL OF MULTIPATH FADING

In a multipath fading channel, multipath components can be divided in two groups [10]: a large number of components which satisfy the conditions of central limit theorem (CLT), and a small number of components, which don't satisfy CLT conditions. Usually, the first group constitutes a Gaussian random process (with Rayleigh envelope PDF), while the second group results in a non-Gaussian random process (with non-Rayleigh envelope PDF).

Specifically, if the signal $x_{tr}(t) = \cos \omega t$ is transmitted through a multipath fading channel, then the received signal has the following form:

$$x_{re}(t) = \sum_{i=1}^{N} A_i \cos(\omega t + \Phi_i) + n(t) \quad (1)$$

where N is the number of components, which don't satisfy CLT conditions, A_i and Φ_i are the amplitude and phase of the ith CLT-violating component, ω is the angular frequency, and $n(t)$ arises from the superposition of a large number of components which satisfy CLT conditions.

In the above formula, N is an arbitrary deterministic constant, A_i's are arbitrary positive dependent random variables, Φ_i's are iid random variables with uniform distributions in $[0, 2\pi]$, and $n(t)$ is a stationary zero-mean Gaussian stochastic process with unit variance (for simplicity), independent of all A_i's and Φ_i's. The assumption of uniform distribution for Φ_i's holds in a large number of situations ([1, pp. 115-116], [11, p. 505]); and can be related to the stationarity concept ([12]-[13]). The Gaussian assumption for $n(t)$ is a direct consequence of CLT. It should be mentioned that the results of this paper can be used exactly for N different ω_i's in (1) ([14]-[15]), instead of a single ω. However, to simplify the derivation of results, a single ω is considered.

III. INTEGRAL FORM OF THE ENVELOPE PDF

The sum of N cosine waves in (1) can be replaced by a single cosine wave of the same angular frequency ω, but with new amplitude and phase (A and Φ). So (1) can be rewritten as:

$$x_{re}(t) = A\cos(\omega t + \Phi) + n(t) = R(t)\cos(\omega t + \Theta(t)) \quad (2)$$

where $R(t)$ and $\Theta(t)$ are defined based on A, Φ, and the inphase and quadrature components of $n(t)$.

Based on the theorem 1 of [16], which characterizes the structure of any spherically symmetric random vector, it is proved in [17] that A and Φ are independent, A is distributed according to the following PDF:

$$f_A(a) = a\int_0^{\infty} \lambda J_0(a\lambda)\Lambda(\lambda)d\lambda = aH_{0a}\{\Lambda(\lambda)\} \quad (3)$$

and Φ is distributed uniformly. In the above formula, $J_0(.)$ is the Bessel function of order zero, and $\Lambda(\lambda)$ is a characteristic function, with the following form:

$$\Lambda(\lambda) = E_{A_1 \ldots A_N}[\prod_{i=1}^{N} J_0(A_i\lambda)] \quad (4)$$

where E denotes mathematical expectation. Moreover, $H_{0a}\{\Lambda(\lambda)\}$ is the zero order Hankel transform of $\Lambda(\lambda)$ [18]; or the so called Fourier-Bessel transform by some authors [19].

In a similar manner [17], it can be shown that for a fixed instant of time $t = t'$, the random variables $R(t')$ and $\Theta(t')$, or briefly R and Θ, are independent, R is distributed according to the following PDF:

Table I
A brief review of previous studies

Method	References
Numerical Integration	24, 26, 28, 31, 32, 34, 38, 43
Fourier-Bessel Series	14, 25, 33, 35
Laguerre Series	15, 25, 26, 28, 30, 35, 39, 40, 41, 42
Analytic Approximation	1 p. 135, 27, 29 p. 15 & 25, 34, 37, 41, 43, 44
Power Series	35, 39
Recursive	36, 40

$$f_R(r) = r\int_0^{\infty} \lambda J_0(r\lambda)\exp(-\lambda^2/2)\Lambda(\lambda)d\lambda$$
$$= r\,H_{0r}\{\exp(-\lambda^2/2)\Lambda(\lambda)\} \quad (5)$$

and Θ is distributed uniformly.

In this paper, attention is paid to $f_R(r)$ as the envelope PDF. In other words, $n(t) \neq 0$. However, in some applications, $n(t) = 0$. Thus, $f_A(a)$ should be considered as the envelope PDF, which is also discussed in [17].

IV. REVIEW OF PREVIOUS RESULTS

For $N = 0$, $f_R(r)$ reduces to the Rayleigh PDF [20]. Assuming $N = 1$, there are several cases in the literature: deterministic A_1 results in the Rice PDF [20], and $f_R(r)$'s obtained by assuming K, lognormal, and Nakagami distributions for A_1 are discussed in [21]-[23], respectively. The behavior of $f_R(r)$ for small and large values of r, assuming an arbitrary distributed A_1, is also discussed in [1, p. 126]. However, for $N \geq 2$ and arbitrary joint distributions of A_i's, it seems impossible to express $f_R(r)$ in terms of the known mathematical functions. Thus, two infinite series, a Laguerre series and a power series, are derived and compared in this paper.

In Table I, a brief survey of previous works for $N \geq 2$ is reported. The pioneering works of Rayleigh, Pearson, Kluyver, and Markov on random vector problem is not included in Table I; because their papers were not available to the authors. It seems that the case of arbitrary dependent A_i's has not been studied previously.

V. POWER AND LAGUERRE SERIES FOR $f_R(r)$

Based on the properties of A, Φ, and $n(t)$ in (2), mentioned earlier in Section III, R, conditioned on A, has a Rice PDF:

$$f_{R|A}(r|a) = r\exp(-(r^2+a^2)/2)I_0(ar) \tag{6}$$

where $I_0(.)$ is the modified Bessel function of order zero. Thus $f_R(r)$ can be written as:

$$f_R(r) = r\exp(-r^2/2)E_A[\exp(-A^2/2)I_0(Ar)] \tag{7}$$

The following generating function for the Laguerre polynomials is given in [45]:

$$\exp(s)J_0(2\sqrt{ys}) = \sum_{n=0}^{\infty}\frac{L_n(y)s^n}{n!} \tag{8}$$

where $L_n(.)$ is the Laguerre polynomial of order n. For $y = A^2/\beta$ and $s = -\beta r^2/4$, it yields:

$$I_0(Ar) = \exp(\beta r^2/4)\sum_{n=0}^{\infty}\frac{1}{n!}L_n(\frac{A^2}{\beta})(-\frac{\beta r^2}{4})^n \tag{9}$$

However, $y = -\beta r^2/4$ and $s = A^2/\beta$ give:

$$I_0(Ar) = \exp(-A^2/\beta)\sum_{n=0}^{\infty}\frac{1}{n!}L_n(-\frac{\beta r^2}{4})(\frac{A^2}{\beta})^n \tag{10}$$

In formulas (9) and (10), β is an arbitrary real number. The role of β will be discussed later.

To simplify the subsequent lengthy formulas, consider the following definition:

$$h_n(z) = E_A[\exp(-zA^2)A^{2n}] \tag{11}$$

Now, inserting (9) and (10) into (7) gives the following results:

$$f_R(r) = \sum_{n=0}^{\infty}v_n(\beta)d_n(\beta,r) \tag{12}$$

$$f_R(r) = \sum_{n=0}^{\infty}w_n(\beta)g_n(\beta,r) \tag{13}$$

where $v_n(\beta)$, $d_n(\beta,r)$, $w_n(\beta)$, and $g_n(\beta,r)$ are:

$$v_n(\beta) = \frac{1}{n!}E_A[\exp(-A^2/2)L_n(\frac{A^2}{\beta})] \tag{14}$$

$$d_n(\beta,r) = r\exp(-(1/2-\beta/4)r^2)(-\frac{\beta r^2}{4})^n \tag{15}$$

$$w_n(\beta) = \frac{1}{n!\beta^n}h_n(\frac{1}{2}+\frac{1}{\beta}) \tag{16}$$

$$g_n(\beta,r) = r\exp(-r^2/2)L_n(-\frac{\beta r^2}{4}) \tag{17}$$

Note that (12) is the power series expansion for $f_R(r)$; while (13) is the Laguerre series expansion.

When the behavior of A_i's is such that the domain of A becomes $[0,\infty[$, direct evaluation of the mathematical expectation in (11) using (3), simplifies $h_n(z)$ to the following form:

$$h_n(z) = \frac{n!}{z^n}\int_0^{\infty}\exp(-x)L_n(x)\Lambda(\sqrt{4zx})dx \tag{18}$$

Otherwise, $h_n(z)$ can't be simplified analytically, and so, should be evaluated by a numerical integration method. A slightly different form of (18) has been appeared in another application [46]. For $z = 1/2$, (18) simplifies to [35, (24)], without confirming on the fact that in using (18), the domain of A should be $[0,\infty[$.

Utilization of the conditional Rice PDF idea, and then expanding $I_0(.)$, as done in (6) and (7), has been used previously ([15], [30], [35], [39]). In fact, for $\beta = 0$, (12) reduces to [35, (23)]; and for $\beta = -2$, (13) simplifies to [15, p. 569], [30, (13)], and [35, (19)]. The main advantage of a variable β, instead of a predetermined constant, lies in the fact that β can be selected in such a way to minimize the truncation error of (12) and (13). This topic is discussed in the following sections in details.

VI. ANALYZING THE TRUNCATION ERRORS

Using just $n_{\max}$ terms in (12) and (13), instead of an infinite number of terms, introduces a truncation error for each series. In what follows, upper bounds for the truncation error of (12) and (13) are presented, with proofs given in [17]. Note that these upper bounds are uniform, i.e. they are independent of *r*. These upper bounds are applicable only when $N > 2$, and also when the maximum value of A can be bounded by a finite positive constant $a_{\max}$. The upper bounds for $N = 2$ are presented completely in [17] and partly in [47], but are not presented in this paper.

When all of the A_i's are independent and bounded random variables (with finite $\max(A_i)$'s), then A becomes a bounded random variable; because $\max(A) = \sum_{i=1}^{N}\max(A_i)$. Note that this case includes deterministic A_i's, since $A_i = \max(A_i)$. When A_i's are bounded but dependent random variables, then A becomes a bounded random variable again, having $\max(A) = \sum_{i=1}^{N}\max(A_i)$. Thus, for the general case of a mixture of deterministic, independent-bounded, and dependent-bounded A_i's, a good selection for $a_{\max}$ is:

$$a_{\max} = \sum_{i=1}^{N}\max(A_i) \tag{19}$$

Deriving upper bounds which are not restricted to bounded random variables, is under study.

For the power series, the uniform upper bound of truncation error in (12) is:

$$\left|E_{n_{\max}}\right|_{uub} = \sum_{n=n_{\max}+1}^{\infty}t_n(\beta)s_n(\beta);\ \beta < 2 \tag{20}$$

where $t_n(\beta)$ and $s_n(\beta)$ are the upper bound and uniform upper bound of $|v_n(\beta)|$ and $|d_n(\beta,r)|$, respectively:

$$t_n(\beta) = \frac{QB}{n!} L_n(\frac{a_{\max}{}^2}{-|\beta|}); \ -\infty < \beta < \infty \tag{21}$$

$$s_n(\beta) = \frac{2}{\sqrt{|\beta|}} (\frac{|\beta|}{2-\beta} \frac{2n+1}{2e})^{n+1/2}; \ \beta < 2 \tag{22}$$

Note that (20) is convergent for the given range of β. The constants Q and B in (21) are given by:

$$Q = \sqrt{a_{\max}{}^3/3} \int_0^\infty \lambda |\Lambda(\lambda)| d\lambda \tag{23}$$

$$B = \begin{cases} [\sqrt{2\pi} a_{\max} \operatorname{erf}(\sqrt{2} a_{\max})/4]^{1/4}; a_{\max} \geq 1 \\ \min([\sqrt{2\pi} a_{\max} \operatorname{erf}(\sqrt{2} a_{\max})/4]^{1/4}, \sqrt{a_{\max}}); a_{\max} \end{cases} \tag{24}$$

where $\operatorname{erf}(.)$ is the error function [45].

For the Laguerre series, the uniform upper bound of truncation error in (13) is:

$$\left|E_{n_{\max}}\right|_{uub} = \sum_{n=n_{\max}+1}^{\infty} q_n(\beta) u_n(\beta); \ \beta \neq 0 \tag{25}$$

where $q_n(\beta)$ and $u_n(\beta)$ are the upper bound and uniform upper bound of $|w_n(\beta)|$ and $|g_n(\beta, r)|$, respectively:

$$q_n(\beta) = \frac{1}{n! |\beta|^n} b_n(\frac{1}{2} + \frac{1}{\beta}); \ \beta \neq 0 \tag{26}$$

$$u_n(\beta) = \begin{cases} \sqrt{(1-2/\beta)2n} \frac{\exp((2/\beta - 1)n)}{n!} (-\frac{\beta n}{2})^n; \beta \leq - \\ \frac{1}{\sqrt{e}}; -4 < \beta \leq 0 \\ \sqrt{\frac{2\beta n}{\beta+2}} \exp(-\beta n/(\beta+2)) L_n(-\frac{\beta^2 n}{2(\beta+2)}); \beta \end{cases} \tag{27}$$

Note that (25) is convergent for the given range of β. The function $b_n(z)$ in (26) is an upper bound for $h_n(z)$ in (11), given by:

$$b_n(z) = \begin{cases} Q \exp(-a_{\max}{}^2 z) \frac{a_{\max}{}^{2n+1/2}}{\sqrt{4n+1}}; z \leq 0 \\ Q \frac{a_{\max}{}^{2n+1/2}}{\sqrt{4n+1}}; z > 0 \end{cases} \tag{28}$$

VII. MINIMIZING THE TRUNCATION ERRORS

Considering the power series, and for a fixed $n_{\max}$, $\beta = 0$ minimizes (20) [17]. So (20) changes to the following form:

$$\left|E_{n_{\max}}\right|_{uub} = \frac{QB}{\sqrt{e}} \sum_{n=n_{\max}+1}^{\infty} \frac{(2n+1)^{n+1/2}}{(n!)^2} (\frac{a_{\max}{}^2}{4e})^n \tag{29}$$

In addition, (12) simplifies to:

$$f_R(r) = r \exp(-r^2/2) \sum_{n=0}^{\infty} \frac{h_n(0.50)}{4^n (n!)^2} r^{2n} \tag{30}$$

On the other hand, considering the Laguerre series, and for a fixed $n_{\max}$, $\beta = -3.9$ minimizes (25) [17]. So (25) changes to the following form:

$$\left|E_{n_{\max}}\right|_{uub} = \frac{Q\sqrt{a_{\max}}}{\sqrt{e}} \sum_{n=n_{\max}+1}^{\infty} \frac{1}{n! \sqrt{4n+1}} (\frac{a_{\max}{}^2}{3.9})^n \tag{31}$$

In addition, (13) simplifies to:

$$f_R(r) = r \exp(-r^2/2) \sum_{n=0}^{\infty} \frac{h_n(0.24)}{(-3.9)^n n!} L_n(0.98 r^2) \tag{32}$$

VIII. COMPARISON OF TWO OPTIMUM SERIES

It can be shown that [17]:

$$\lim_{n\to\infty} \frac{t_{n+1}(0) s_{n+1}(0)}{t_n(0) s_n(0)} = \frac{a_{\max}{}^2}{2} \lim_{n\to\infty} \frac{1}{n} = 0 \tag{33}$$

$$\lim_{n\to\infty} \frac{q_{n+1}(-3.9) u_{n+1}(-3.9)}{q_n(-3.9) u_n(-3.9)} = \frac{a_{\max}{}^2}{3.9} \lim_{n\to\infty} \frac{1}{n} = 0 \tag{34}$$

Clearly, the convergence rate of (29) is approximately one half of the (31). From this point of view, (30) is inferior to (32). In other words, the optimum Laguerre series is more efficient than the optimum power series, considering convergence rate as the desired criterion.

IX. CONCLUSION

In this paper, a general multipath fading model is introduced in (1). The envelope PDF of this model can be expressed by (5). However, to obtain formulas which are more appropriate for both analytic and numerical purposes, two infinite expansions are reported in (12) and (13). After minimizing the upper bounds for the truncation error of these two infinite expansions, It is shown that the optimum Laguerre series is more efficient than the optimum power series. Based on (31), and for a given number of significant digits, (5) can be approximated by (32) efficiently, with a minimum number of terms.

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