A General Framework for the Characterization of Multipath Fading

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Abstract -- **To incorporate the effect of multipath fading in system analysis and design, we need the probability density function (PDF) of the received signal envelope. In this paper, we consider a general multipath fading channel with arbitrary number of paths, where the amplitudes of multipath components are arbitrary correlated positive random variables. Since the integral-form of the envelope PDF in such a general setting is not easy to manipulate, we develop a Laguerre series for the PDF. A useful tight uniform bound on the truncation error of the Laguerre series is also derived. This Laguerre series expresses the envelope PDF just in terms of simple polynomial-exponential kernels. Hence, it is particularly useful for mathematical performance analysis of wireless systems in those multipath propagation environments, where the number of strong multipath components is relatively small. We demonstrate the utility of the Laguerre series by deriving a generic closed-form expression for the average bit error rate of an arbitrary modulation scheme in multipath fading channels.**

I. INTRODUCTION

When an unmodulated carrier is transmitted in a multipath fading channel, it breaks into several multipath components. These components add vectorially, according to their amplitudes and phases, and a fluctuating envelope is experienced by the receiver. In this paper, we study the envelope probability density function (PDF), using the random vector model, assuming that the number of multipath components is an arbitrary constant, phases are independent with uniform distributions, and the amplitudes are correlated positive random variables with arbitrary distributions. This setting is general enough to cover many cases of interest.

II. A GENERAL MODEL FOR MULTIPATH FADING

In a multipath fading channel, multipath components can be divided into two independent groups: the first group consists of a small number of strong multipath components which do not satisfy the conditions of the central limit theorem (CLT), while the second group contains a large number of weak multipath components which satisfy the CLT conditions. The first group generates a non-Gaussian random process with non-Rayleigh envelope PDF, while the second group results in a Gaussian random process with the Rayleigh envelope PDF. Specifically, if the signal $\vartheta_{tr}(t') = \cos 2\pi f_c t'$ is transmitted through the channel, where f_c is the carrier frequency and t' is the time, then the received signal is:

$$\vartheta_{re}(t') = \vartheta(t') + \hat{\vartheta}(t') , \quad (1)$$

where $\vartheta(t')$ and $\hat{\vartheta}(t')$ are independent processes defined by:

$$\vartheta(t') = \sum_{i=1}^{N} A_i \cos(2\pi f_c t' + \Phi_i) , \quad (2)$$

$$\hat{\vartheta}(t') = \sum_{i=1}^{\hat{N}} \hat{A}_i \cos(2\pi f_c t' + \hat{\Phi}_i) . \quad (3)$$

In the above formulas, N and $\hat{N}$ are the number of multipath components in the first and the second groups, respectively, A_i 's and $\hat{A}_i$'s represent the amplitudes of multipath components, while Φ_i 's and $\hat{\Phi}_i$'s stand for the phases of multipath components. In the first group, N is an arbitrary constant, A_i 's are dependent positive random variables with arbitrary distributions, independent of Φ_i 's, and Φ_i 's are independent of each other with uniform distributions on $[0, 2\pi)$. The signal $\vartheta(t')$ generated by the first group is a non-Gaussian random process with a non-Rayleigh envelope. In the second group, $\hat{N}$ is large enough and the statistical properties of $\hat{A}_i$'s and $\hat{\Phi}_i$'s are such that based on CLT, $\hat{\vartheta}(t')$ can be modeled as a Gaussian random process with the Rayleigh envelope.

Now we give a brief justification for the statistical properties of Φ_i 's, which were stated above. The phase of each multipath component, with respect to an arbitrary but fixed reference, depends on its path length. It changes by 2π as its associated path length changes by a wavelength. If there are large variations among the path lengths (which is usually the case), the phases of multipath components vary over a wide range as well. These phases, when reduced to module 2π, can be accurately modeled by random variables with uniform distributions on $[0, 2\pi)$ [1] [2]. Large variations among the path lengths also indicate that the phases of multipath components can be reasonably considered as independent random variables. It is interesting to note that there is a close relationship between uniform distribution of phases and the concept of stationarity [3] [4].

To develop the random vector model for our multipath fading channel, we note that the sums of cosine waves in (2) and (3) can be replaced by the following single cosine waves:

$$\vartheta(t') = A\cos(2\pi f_c t' + \Phi) , \quad (4)$$

$$\hat{\vartheta}(t') = \hat{A}\cos(2\pi f_c t' + \hat{\Phi}) . \quad (5)$$

These representations lead to this representation for $\vartheta_{re}(t')$ in (1):

$$\vartheta_{re}(t') = R\cos(2\pi f_c t' + \Theta) . \quad (6)$$

Clearly, the pairs (A, Φ) and ($\hat{A}$, $\hat{\Phi}$) represent the envelopes and the phases of signals from the first and the second groups of multipath components, respectively, while R and Θ are the envelope and phase of the signal composed of the two types of multipath components. The interrelationships among the random variables A, Φ, $\hat{A}$, $\hat{\Phi}$, R, and Θ can be easily understood by using the vector notation:

$$A\exp(j\Phi)=\sum_{i=1}^{N}A_i\exp(j\Phi_i)\,,$$

$$\hat{A}\exp(j\hat{\Phi})=\sum_{i=1}^{\hat{N}}\hat{A}_i\exp(j\hat{\Phi}_i)\,,$$

$$R\exp(j\Theta)=A\exp(j\Phi)+\hat{A}\exp(j\hat{\Phi})\,, \tag{7}$$

where $j=\sqrt{-1}$. Now we define X, Y, $\hat{X}$, $\hat{Y}$, V, and W:

$$X=A\cos\Phi,\quad Y=A\sin\Phi\,,$$

$$\hat{X}=\hat{A}\cos\hat{\Phi},\quad \hat{Y}=\hat{A}\sin\hat{\Phi}\,,$$

$$V=R\cos\Theta,\quad W=R\sin\Theta\,, \tag{8}$$

which we need in the next section for deriving the PDF's of A, Φ, $\hat{A}$, $\hat{\Phi}$, R and Θ.

III. Distributions of A, Φ, $\hat{A}$, $\hat{\Phi}$, R, and Θ

A. *PDF's of A and* Φ

Based on (7) - (8), X and Y are functions of A_i 's and Φ_i 's, i.e.:

$$X=\sum_{i=1}^{N}A_i\cos\Phi_i,\quad Y=\sum_{i=1}^{N}A_i\sin\Phi_i\,. \tag{9}$$

So, the joint characteristic function (CF) of X and Y, defined by $\Psi_{XY}(\eta,\zeta)=E_{XY}[\exp(j\eta X+j\zeta Y)]$, can be written as:

$$\begin{aligned}\Psi_{XY}(\eta,\zeta)&=E_{A_1\ldots A_N\Phi_1\ldots\Phi_N}[\exp(j\eta X+j\zeta Y)]\,,\\&=E_{A_1\ldots A_N}[E_{\Phi_1\ldots\Phi_N}[\exp(j\eta X+j\zeta Y)\,|\,A_1\ldots A_N]]\,.\end{aligned} \tag{10}$$

Since Φ_i 's are independent of A_i 's, we can change the conditional expectation in (10) to an ordinary expectation:

$$\Psi_{XY}(\eta,\zeta)=E_{A_1\ldots A_N}[E_{\Phi_1\ldots\Phi_N}[\exp(j\eta X+j\zeta Y)]]\,. \tag{11}$$

Using (9) we obtain:

$$\Psi_{XY}(\eta,\zeta)=E_{A_1\ldots A_N}[E_{\Phi_1\ldots\Phi_N}[\prod_{i=1}^{N}\exp(j\eta A_i\cos\Phi_i+j\zeta A_i\sin\Phi_i)]]\,, \tag{12}$$

which can be simplified based on the fact that Φ_i 's are independent of each other:

$$\Psi_{XY}(\eta,\zeta)=E_{A_1\ldots A_N}[\prod_{i=1}^{N}E_{\Phi_i}[\exp(j\eta A_i\cos\Phi_i+j\zeta A_i\sin\Phi_i)]]\,. \tag{13}$$

If we define the polar variables λ and ω in terms of the rectangular variables η and ζ:

$$\eta=\lambda\cos\omega,\quad \zeta=\lambda\sin\omega\,, \tag{14}$$

then the exponent in (13) can be simplified to:

$$\eta A_i\cos\Phi_i+\zeta A_i\sin\Phi_i=A_i\lambda\cos(\Phi_i-\omega)\,. \tag{15}$$

By inserting (15) into (13) we obtain:

$$\Psi_{XY}(\eta,\zeta)=E_{A_1\ldots A_N}[\prod_{i=1}^{N}\int_0^{2\pi}\exp(A_i\lambda\cos(\phi_i-\omega))\,f_{\Phi_i}(\phi_i)\,d\phi_i]\,. \tag{16}$$

As mentioned earlier, Φ_i 's have uniform distributions on $[0,2\pi)$, i.e. $f_{\Phi_i}(\phi_i)=1/2\pi$. So, according to the integral form of $J_0(.)$, the zero order Bessel function:

$$J_0(z)=(2\pi)^{-1}\int_0^{2\pi}\exp(jz\cos\xi)\,d\xi\,, \tag{17}$$

we can simplify (16) to:

$$\Psi_{XY}(\eta,\zeta)=E_{A_1\ldots A_N}[\prod_{i=1}^{N}J_0(A_i\lambda)]=\Lambda(\lambda)\,. \tag{18}$$

According to the definition of the jointly spherically symmetric random variables [5], we observe that X and Y are jointly spherically symmetric as $\Psi_{XY}(\eta,\zeta)$ in (18) is only a function of $\lambda=\sqrt{\eta^2+\zeta^2}$. Hence, based on Theorem 1 in [5], it can be inferred that A and Φ are independent variables with the following PDF's:

$$f_A(a)=a\int_0^{\infty}\lambda J_0(a\lambda)\Lambda(\lambda)\,d\lambda=aH_{0a}\{\Lambda(\lambda)\}\,, \tag{19}$$

$$f_\Phi(\phi)=1/2\pi\,. \tag{20}$$

Obviously, A is a non-Rayleigh variable. In Eq. (19), $H_{0z}\{G(\xi)\}$ is the zero order Hankel transform of $G(.)$, defined by [6]:

$$H_{0z}\{G(\xi)\}=\int_0^{\infty}\xi J_0(z\xi)G(\xi)\,d\xi\,. \tag{21}$$

This integral transform has also been referred to as the Fourier-Bessel transform [7] [8]. Based on the extensive tables in [9], one can calculate the Hankel transform of a large number of functions. It should be mentioned that not only the PDF of A, but also the PDF of an arbitrary positive random variable P can be expressed in a form similar to (19) [10], where for P we have $\Lambda(\lambda)=E_P[J_0(P\lambda)]$. For the random variable A, the corresponding $\Lambda(\lambda)$ is given in (18).

Eq. (19) was originally presented in [11] and [12], with A_i 's as nonrandom constants. The random walk problem associated with (19) was first addressed in [13] and [14]. More details and information about $f_A(a)$ in (19) can be found in [15].

B. *PDF's of* $\hat{A}$ *and* $\hat{\Phi}$

As stated in Section II, we assume that CLT holds for $\hat{\vartheta}(t')$ in (3). By this assumption we imply that $\hat{N}$ is a large constant and the statistical properties of $\hat{A}_i$'s and $\hat{\Phi}_i$ ' s are such that $\hat{X}$ and $\hat{Y}$ are two independent zero-mean Gaussian random variables with the same variance $\hat{\sigma}^2$. In the sequel and without loss of generality, let $\hat{\sigma}^2=1$, to simplify the notation. The CF's of $\hat{X}$ and $\hat{Y}$ are:

$$\Psi_{\hat{X}}(\eta)=\Psi_{\hat{Y}}(\eta)=\exp(-\eta^2/2)\,. \tag{22}$$

Hence, the joint CF of $\hat{X}$ and $\hat{Y}$, $\Psi_{\hat{X}\hat{Y}}(\eta,\zeta)$, can be easily derived:

$$\Psi_{\hat{X}\hat{Y}}(\eta,\zeta)=\exp(-\lambda^2/2)\,. \tag{23}$$

We observe that $\hat{X}$ and $\hat{Y}$ are jointly spherically symmetric, since $\Psi_{\hat{X}\hat{Y}}(\eta,\zeta)$ in (23) is only a function of $\lambda=\sqrt{\eta^2+\zeta^2}$. So, based on Theorem 1 in [5], it can be concluded that $\hat{A}$ and $\hat{\Phi}$ are independent random variables with the following PDF's:

$$f_{\hat{A}}(\hat{a})=\hat{a}\exp(-\hat{a}^2/2)\,, \tag{24}$$

$$f_{\hat{\Phi}}(\hat{\phi})=1/2\pi\,. \tag{25}$$

Clearly, $\hat{A}$ is a Rayleigh variable with unit mode.

C. *PDF's of R and* Θ

By combining (7) and (8) we obtain:

$$V=X+\hat{X},\quad W=Y+\hat{Y}\,. \tag{26}$$

Since X and Y are independent of $\hat{X}$ and $\hat{Y}$, the joint CF of V and W, $\Psi_{VW}(\eta,\zeta)$, can be written as:

$$\Psi_{VW}(\eta,\zeta)=\Psi_{XY}(\eta,\zeta)\Psi_{\hat{X}\hat{Y}}(\eta,\zeta)\,. \tag{27}$$

By substituting (18) and (23) into (27) we obtain:

$$\Psi_{VW}(\eta,\zeta)=\Lambda(\lambda)\exp(-\lambda^2/2)\,. \tag{28}$$

Similar to the previous two cases, V and W are jointly spherically symmetric. So, R and Θ are independent with the following PDF's:

$$f_R(r)=r\int_0^{\infty}\lambda J_0(r\lambda)\exp(-\frac{\lambda^2}{2})\Lambda(\lambda)\,d\lambda=rH_{0r}\{\exp(-\frac{\lambda^2}{2})\Lambda(\lambda)\} \tag{29}$$

$$f_\Theta(\theta)=1/2\pi\,. \tag{30}$$

The PDF in (29) completely characterizes the statistical behavior of the envelope in the multipath fading model of Section II.

However, the integral-form of (29) is not convenient for analytic calculations, which are essential for mathematical analysis of wireless communication systems over multipath fading channels. So, in the next section we review other alternatives for representing $f_R(r)$, which may be more suitable for analytic studies.

IV. Review of the Previous Results on $f_R(r)$

For $N=0$ and according to (7) and (24), $f_R(r)$ reduces to Rayleigh PDF which was first introduced in [16]. For $N=1$ and A_1 as a constant, $f_R(r)$ becomes the Rice PDF [17]. When $N=1$ and A_1 is a random variable distributed according to lognormal, Nakagami, generalized Bessel, or *K* distribution, the corresponding $f_R(r)$ is derived in [1], [18], [19], and [20], respectively. The behavior of $f_R(r)$ over small and large values of *r*, for $N=1$ and an arbitrary PDF for A_1, is discussed in [1].

For $N \geq 2$ and different presumptions on the distributions of A_i's, the following methods are used for manipulating $f_R(r)$:

1) *Numerical calculation of the integral in* (29) [21]-[25],
2) *Infinite expansion in terms of Fourier-Bessel series* [26] [27],
3) *Infinite expansion in terms of Laguerre series* [24] [27]-[30],
4) *Different analytic approximations* [22]-[24] [31],
5) *Infinite expansion in terms of the power series* [27],
6) *Infinite expansion using hypergeometric series* [24] [32]-[34],
7) *Miscellaneous* [24].

Among the above methods, only those listed in 3), 5), 6), and 7) provide exact solutions for $f_R(r)$. Method 6) is not suitable for numerical calculations, while method 7) has been developed only for $N=2$. Hence, we focus on 3) and 5), which are exact and also appropriate for both numerical and analytic calculations.

In Section V we present two infinite series for $f_R(r)$, a Laguerre series and a power series. Tight upper bounds on the truncation error of the Laguerre series and the power series are derived in [35]. By comparing the two truncation error results, it is shown in [35] that for a fixed number of terms, the truncated Laguerre series provides a smaller truncation error. So, from this point of view, Laguerre series is preferred to the power series for the envelope PDF. Note that the infinite series of other references listed in 3) and 5) are special cases of those reported here. Moreover, the truncation errors of Laguerre and power series are not analyzed in other references.

V. Series Expansions of $f_R(r)$

Based on (7), the conditional PDF $f_{R|A}(r \mid a)$ is Rician [1]:

$$f_{R|A}(r \mid a) = r\exp[-(r^2+a^2)/2] I_0(ar), \quad (31)$$

where $I_0(.)$ is the zero order modified Bessel function. After averaging with respect to *A* we obtain:

$$f_R(r) = E_A[f_{R|A}(r|a)] = r\exp(-r^2/2) E_A[\exp(-A^2/2) I_0(Ar)]. \quad (32)$$

Now we take this Laguerre polynomial generating function [36]:

$$\exp(\sigma) J_0(2\sqrt{\tau\sigma}) = \sum_{n=0}^{\infty} L_n(\tau)\sigma^n/n!, \quad (33)$$

where $L_n(.)$ is the Laguerre polynomial of order *n*. For $\tau = -\beta r^2/4$ and $\sigma = A^2/\beta$, (33) changes to:

$$I_0(Ar) = \exp(-A^2/\beta)\sum_{n=0}^{\infty} L_n(-\beta r^2/4)(A^2/\beta)^n/n!, \quad \beta \neq 0, \quad (34)$$

while for $\tau = A^2/\beta$ and $\sigma = -\beta r^2/4$, (33) modifies to:

$$I_0(Ar) = \exp(\beta r^2/4)\sum_{n=0}^{\infty} L_n(A^2/\beta)(-\beta r^2/4)^n/n!, \quad \beta \neq \pm\infty. \quad (35)$$

Formulas (34) and (35) may be considered as parametric expansions for $I_0(Ar)$ where the parameter β is a real number, non-zero for (34) and finite in (35). For $\beta = \pm\infty$ in (34) and $\beta = 0$ in (35), we obtain the Maclaurin series of $I_0(.)$:

$$I_0(Ar) = \sum_{n=0}^{\infty} (Ar/2)^{2n}/(n!)^2. \quad (36)$$

To proceed further, we need to define the following function:

$$h_n(z) = E_A[\exp(-zA^2)A^{2n}], \quad (37)$$

where *z* is a real number and the PDF of *A* is given in (19). This function is discussed in details in [35].

By replacing $I_0(Ar)$ in (32) with (34) and (35), we obtain the Laguerre and the power series for $f_R(r)$, respectively:

$$f_R(r) = \sum_{n=0}^{\infty} w_n(\beta) g_n(\beta, r), \quad \beta \neq 0, \quad (38)$$

$$f_R(r) = \sum_{n=0}^{\infty} v_n(\beta) d_n(\beta, r), \quad \beta \neq \pm\infty, \quad (39)$$

in which:

$$w_n(\beta) = h_n(2^{-1}+\beta^{-1})/(n!\beta^n), \quad (40)$$

$$g_n(\beta, r) = r\exp(-r^2/2) L_n(-\beta r^2/4), \quad (41)$$

$$v_n(\beta) = E_A[\exp(-A^2/2) L_n(A^2/\beta)]/n!, \quad (42)$$

$$d_n(\beta, r) = r\exp[-(\tfrac{1}{2}-\tfrac{\beta}{4})r^2](-\beta r^2/4)^n. \quad (43)$$

In order to calculate the coefficient $w_n(\beta)$ in (38), only $h_n(2^{-1}+\beta^{-1})$ is required. However, for computing the coefficient $v_n(\beta)$ in (39), we need $h_0(1/2)$, ..., $h_n(1/2)$, since based on [36]:

$$L_n(z) = \sum_{k=0}^{n} (-1)^k n! z^k/[(n-k)!(k!)^2], \quad (44)$$

Eq. (42) can be written as:

$$v_n(\beta) = \sum_{k=0}^{n} (-1)^k h_k(1/2)/[(n-k)!(k!)^2\beta^k]. \quad (45)$$

Application of the conditional Rice PDF in (31) and replacement of $I_0(.)$ in (32) with an infinite series is also discussed in [27] [29] [30]. The advantage of this approach, in comparison with expanding $f_R(r)$ directly in terms of an infinite series, is that it provides a non-Rayleigh but perturbed-Rayleigh PDF. In other words, we usually expect a Rayleigh PDF for $f_R(r)$ in a multipath channel. In the presence of several strong multipath components that violate the CLT, $f_R(r)$ deviates from Rayleigh PDF. The amount of this deviation can be easily quantified in terms of the statistical properties of the amplitudes of these CLT-violating multipath components, if we use the conditional-Rice-PDF approach. This idea will be further explored in the next section. The conditional-Rice-PDF method is similar to the common approach for expanding non-Gaussian PDF's using Gram-Charlier type series [37] [38].

In previous works, only fixed values have been considered for β. Specifically, Eq. (38) is used in [27] [29] [30] with $\beta = -2$, while Eq. (39) is employed in [27] assuming $\beta = 0$. The advantage of considering β as a variable parameter lies on the fact that β can be optimized in such a way to minimize the truncation errors of (38) and (39). In fact, it is proved in [35] that $\beta = -4$ and $\beta = 0$ minimize the truncation errors of (38) and (39), respectively. Moreover, it is shown that the convergence rate of the Laguerre series is twice. Hence, based on these observations, we choose the Laguerre series in (38) with $\beta = -4$ for the envelope PDF:

$$f_R(r) = r\exp(-r^2/2)\sum_{n=0}^{\infty} h_n(1/4) L_n(r^2)/[(-4)^n n!], \quad r \geq 0, \quad (46)$$

where $h_n(1/4)$ can be computed using any of the methods discussed in [35]. The following tight uniform upper bound is also derived in [35] on the truncation error of (46) with $n_{\max}+1$ terms:

$$\varepsilon = 2.8\,K\,A_{\max}^{1/2} \sum_{n=n_{\max}+1}^{\infty} \sqrt{n}\,(A_{\max}^2/4)^n \left|L_n(4n)\right| \exp(-2n)/n!\,, \quad (47)$$

where $A_{\max} = \sum_{i=1}^{N} A_{i,\max}$, with $A_{i,\max} < \infty$ as the maximum value of the random variable A_i , and also:

$$K = \pi^{-1/2} A_{\max} \int_0^{\infty} \sqrt{\lambda}\,\left|\Lambda(\lambda)\right| d\lambda\,. \quad (48)$$

in which $\Lambda(\lambda)$ is given in (18). Notice (47) is a uniform truncation error, i.e., it holds over the entire range of $0 < r < \infty$.

VI. A New Approach to Bit Error Rate Analysis

So far we have derived a non-Rayleigh but perturbed-Rayleigh representation for the envelope PDF in multipath fading channels. The important "perturbed-Rayleigh" characteristic can be better understood by rewriting (46) as:

$$f_R(r) = h_0(1/4) r \exp(-r^2/2)$$
$$+\sum_{n=1}^{\infty} h_n(1/4)(-1)^n r \exp(-r^2/2) L_n(r^2)/(4^n n!), \quad r \geq 0. \quad (49)$$

Notice that [39]:

$$\int_0^{\infty} r \exp(-r^2/2) L_n(r^2)\,dr = (-1)^n\,. \quad (50)$$

Moreover, since $f_R(r)$ integrates to 1, (46) yields:

$$\sum_{n=0}^{\infty} h_n(1/4)/(4^n n!) = 1\,. \quad (51)$$

Hence, (49) represents the non-Rayleigh envelope PDF as a mixture (convex combination) of the Rayleigh PDF $r\exp(-r^2/2)$ and the unit-area kernels $(-1)^n r \exp(-r^2/2) L_n(r^2)$, $n = 1, 2, \ldots$. Notice that these kernels are not PDF since they take negative values for some *r*, however, it may be useful to consider them as pseudo PDF's since they integrate to one. When there is no strong multipath component, i.e. $N = 0$, we have $h_0(1/4) = 1$ and $h_1(1/4) = h_2(1/4) = \ldots = 0$, which in turn yield $f_R(r) = r\exp(-r^2/2)$. In the presence of at least one strong multipath component, i.e., $N \geq 1$, we get $0 \leq h_0(1/4) < 1$ while $h_n(1/4) > 0$ for at least one *n* from the set $\{1, 2, \ldots\}$.

The show the insightful role of the mixture representation in (49) for analytic calculations in non-Rayleigh multipath fading channels, we consider the general problem of computing the average bit error rate (BER) of an arbitrary modulation scheme in such channels. Let $P_b(R)$ denote the BER of a particular modulation in the presence of additive white Gaussian noise, conditioned on *R*. As an example, for differential phase shift keying (DPSK) modulation we have $P_b(R) = \exp(-\gamma_b R^2)/2$ [40], where $\gamma_b = E_b/N_0$ represents the signal-to-noise ratio (SNR) per bit, with E_b as the transmitted energy per bit and N_0 as the noise power spectral density. Based on (49), the average BER $\bar{P}_b = E_R[P_b(R)]$ can be written as

$$\bar{P}_b = h_0(1/4)\bar{P}_b^{Rayleigh} + \sum_{n=1}^{\infty} [h_n(1/4)/(4^n n!)]\bar{P}_{b,n}\,, \quad (52)$$

where:

$$\bar{P}_b^{Rayleigh} = \int_0^{\infty} P_b(r)\; r\exp(-r^2/2)\;\; dr\,, \quad (53)$$

$$\bar{P}_{b,n} = \int_0^{\infty} P_b(r)(-1)^n r \exp(-r^2/2) L_n(r^2) dr, \quad n = 0, 1, 2, \ldots\,. \quad (54)$$

Note that $\bar{P}_b^{Rayleigh} = \bar{P}_{b,0}$. Now we observe that the average BER in a non-Rayleigh multipath fading channel can be decomposed into two different parts: the Rayleigh average BER, $\bar{P}_b^{Rayleigh}$, obtained by averaging $P_b(R)$ with respect to the Rayleigh PDF $r\exp(-r^2/2)$, and the pseudo average BER's, $\bar{P}_{b,n}$, derived from averaging $P_b(R)$ with respect to the pseudo PDF's $(-1)^n r\exp(-r^2/2)$ $L_n(r^2)$. For $N = 0$ we have $\bar{P}_b = \bar{P}_b^{Rayleigh}$, while for $N \geq 1$, $\bar{P}_b$ is a convex combination of $\bar{P}_b^{Rayleigh}$ and $\bar{P}_{b,n}$'s, with the weights determined by the statistical characteristics of the amplitudes of strong multipath components A_i . Notice that such a useful representation for $\bar{P}_b$ cannot be obtained by the application of $f_R(r)$ in (29). In fact, for the average BER it gives:

$$\bar{P}_b = \int_0^{\infty} [\int_0^{\infty} P_b(r)\, r\, J_0(r\lambda)\, dr]\, \lambda \exp(-\lambda^2/2) \Lambda(\lambda) d\lambda\,, \quad (55)$$

which cannot be further simplified and in contrast with (52), provides no useful insight.

VII. A Case Study

In this section we consider a numerical example and apply the Laguerre series for calculating the envelope PDF and the average BER of DPSK. Consider a multipath fading channel where in addition to the large number of weak (CLT-satisfying) multipath components, there are four strong (CLT-violating) multipath components: $A_1 = 0.5$, $A_2 = A_3 = 1.5$, and $A_4 = 2.5$. Suppose we want to truncate the Laguerre series in (46) at $n = n_{\max}$ such that $\varepsilon \leq 10^{-4}$. Notice that $A_{\max} = 6$, while based on (18):

$$\Lambda(\lambda) = J_0(0.5\lambda) J_0^{\,2}(1.5\lambda) J_0(2.5\lambda)\,. \quad (56)$$

To determine $n_{\max}$ using (47), we need to calculate the value of *K* in (48), which yields $K = 1.13512$. Based on the numerical values of *K* and $A_{\max}$, it is easy to verify that $n_{\max} = 31$ is the smallest integer that satisfies $\varepsilon \leq 10^{-4}$ according to (47). Now we should compute $h_n(1/4)$ for $n = 0, 1, \ldots, 31$. We have used the definition of $h_n(.)$ in (37). To check the accuracy of the values of $h_n(1/4)$, we observed:

$$\sum_{n=0}^{31} h_n(1/4)/(4^n n!) = 1.00027\,, \quad (57)$$

which in comparison with the identity in (51), shows that our numerical results are fairly accurate.

The truncated Laguerre series in (46) with 32 terms, given by:

$$f_R^{truncated}(r) = r\exp(-r^2/2) \sum_{n=0}^{31} h_n(1/4) L_n(r^2)/[(-4)^n n!]\,, \quad (58)$$

is plotted in Fig. 1. As we expect, it is nonnegative for any *r*. On the other hand, based on (57), the area under (58) is 1.00027 , which is quite acceptable. In Fig. 2 we have plotted the absolute difference between the exact PDF in (29) and the approximate one in (58), i.e. $|\, f_R(r) - f_R^{truncated}(r)\,|$. Inspection of Fig. 2 reveals that over the most of the range of *r*, the truncation error is less than 10^{-4} . Some larger truncation errors around $r = 3$ may be attributed to the loss of precision in numerical calculations. Anyway, as we see in the sequel, the truncation error over the whole range $r \geq 0$ is so small that we can get very accurate numerical results for a typical average BER calculation using $f_R^{truncated}(r)$.

To show the utility of the Laguerre series for analytic manipulations, we derive expressions for the average BER of DPSK in our channel using the approximate and exact envelope PDF's, given in (58) and (29), respectively. For this purpose, first we calculate the integral in (54) and the inner integral in (55) with $P_b(r) = \exp(-\gamma_b r^2)/2$ according to [39]:

$$\int_0^{\infty} \tfrac{1}{2} \exp(-\gamma_b r^2)(-1)^n r \exp(-r^2/2) L_n(r^2) dr = \frac{(1-2\gamma_b)^n}{2(1+2\gamma_b)^{n+1}}, \quad (59)$$

$$\int_0^\infty \tfrac{1}{2}\exp(-\gamma_b r^2) r J_0(\lambda r) dr = \exp[-\lambda^2/(4\gamma_b)]/(4\gamma_b) . \qquad (60)$$

The approximate and exact average BER's can be obtained by substituting (59) and (60) into (52) and (55), respectively:

$$\overline{P}_b^{truncated} = \frac{h_0(1/4)}{2(1+2\gamma_b)} + \sum_{n=1}^{31} \frac{h_n(1/4)}{4^n n!} \frac{(1-2\gamma_b)^n}{2(1+2\gamma_b)^{n+1}} , \qquad (61)$$

$$\overline{P}_b = (4\gamma_b)^{-1} \int_0^\infty \lambda \exp\{-[1+(2\gamma_b)^{-1}]\lambda^2/2\} \Lambda(\lambda) d\lambda , \qquad (62)$$

where in $\overline{P}_b^{truncated}$ only 32 terms are taken into account. Notice that the first term in (61), $1/[2(1+2\gamma_b)]$, is the average BER of DPSK for Rayleigh PDF with unit mode [40].

To check the numerical accuracy of $\overline{P}_b^{truncated}$ in (61), we compared it with $\overline{P}_b$ in (62) over the wide SNR range of $0 \le \gamma_b \le 30$ dB. We observed that $\overline{P}_b^{truncated}$ and $\overline{P}_b$ match up to three significant digits. This good match is not surprising, as we have already shown the high accuracy of $f_R^{truncated}(r)$ in (58) over the entire range $r \ge 0$, regardless of using a finite number of terms.

VIII. Conclusion

In this paper we have considered a general model for multipath fading channels where in addition to a large number of weak multipath components, there are a limited number of strong multipath components. According to a comprehensive literature survey reported in this paper, our model includes many available models as special cases. First we have derived an integral expression for the envelope PDF. Then we have expressed the envelope PDF in terms a Laguerre series, which is more convenient for mathematical analysis. Furthermore, we have derived a formula by which one can easily determine the minimum number of terms in the infinite Laguerre series, which guarantees a specified truncation error over the entire range of the envelope values on the real line. A numerical example is also provided which supports the theoretical results.

In the light of the Laguerre series representation, we have expressed the non-Rayleigh envelope PDF as a mixture of a Rayleigh PDF and non-Rayleigh unit-area kernels (pseudo PDF's). Such a decomposition allows us to study the effects of weak and strong multipath components on the performance of communication systems, separately. As an example, we have shown that the average bit error rate (BER) of any modulation method in multipath fading channels can be decomposed into two separate parts: the first part represents the average BER due to Rayleigh fading, while the second part shows the contribution of the few strong multipath components. Such a representation provides a better understanding of the impact of multipath propagation.

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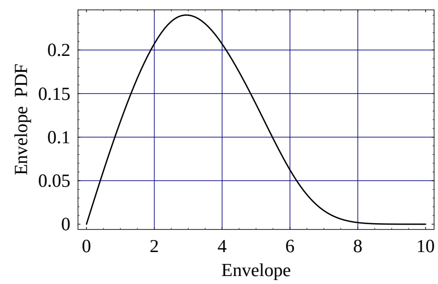

Fig. 1. The envelope PDF in a fading channel with four strong multipath components, calculated using the truncated Laguerre series.

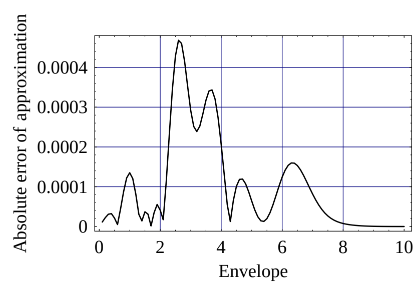

Fig. 2. The absolute difference between the exact and approximate envelope PDF's for the fading channel with four strong multipath components.