Envelope PDF in Multipath Fading Channels with Random Number of Paths and Nonuniform Phase Distributions

ALI ABDI AND MOSTAFA KAVEH

DEPT. OF ELEC. AND COMP. ENG., UNIVERSITY OF MINNESOTA
4-174 EE/CSCI BLDG., 200 UNION ST SE
MINNEAPOLIS, MN 55455, USA

FAX: (612) 625 4583
Email: abdi@ece.umn.edu kaveh@ece.umn.edu

Abstract

In a multipath fading channel, the transmitted signal travels through several different paths to the receiver. In each path, amplitude and phase of the signal vary in a random manner. It is common to consider the number of paths as a large constant and to model random fluctuations of the phase by the uniform probability density function (PDF). However, these assumptions are not realistic in many cases. In this paper, a general multipath fading channel with random number of paths (with negative binomial distribution) and nonuniform phase distributions (with von Mises PDFs) is considered and it is shown that the envelope fluctuates according to a gamma PDF. It is also shown that the parameters of this gamma PDF are directly related to the physical parameters of the channel. Due to the realistic assumptions made in the derivation, the gamma PDF is a promising candidate for accurate modeling of envelope statistics in multipath fading channels.

1-Introduction

When a signal propagates through a multipath environment, it breaks into several multipath components (or briefly, components); and at the receiver, the superposition of these components is observed. In general, the phase of each component, with respect to an arbitrary but fixed reference, depends on its path length. It changes by 2π as its path length changes by a wavelength.

If the dimensions of a cluster of scatterers is much larger than the signal wavelength, there will be large variations among the path lengths and hence the phases of components which are scattered from that cluster. These phases, when reduced to module 2π, can be reasonably modeled as random variables with uniform distributions in $[0,2\pi[$. Based on this assumption, and several other simplifying assumptions, the well-known Rayleigh probability density function (PDF) can be used for the envelope of superimposed components at the receiver [1]. There are several other Rayleigh-based PDFs for the envelope of multipath signals, like the Suzuki distribution [1], the *K* distribution [2], and the distribution which arises due to the presence of a limited number of strong components [3]-[6]. In deriving the above PDFs, the assumption of uniformity for phase distributions plays an important role.

On the other hand, when the signal wavelength is comparable with the dimensions of a cluster of scatterers, the uniform distribution is not suitable for modeling the phases of components. Thus the derivation of the envelope PDF assuming nonuniform distributions for phases is of interest.

To the knowledge of the authors, the effect of nonuniform phase distributions on the envelope PDF has been discussed only in [7] and [8]. Assuming that the number of components (or equivalently, the number of paths) is a large constant such that central limit theorem holds for the in-phase and quadrature components of the received signal, formula (4.6-28) in [7] is derived for the envelope PDF. This complicated formula simplifies to formula (4.6-29) in [7], when the phase variable of each wave is distributed symmetrically about its mean value, and uncorrelated with its associated amplitude variable. Unfortunately, even formula (4.6-29) is too involved to be used in practice.

In this paper, a general and mathematically tractable model for multipath fading channels is considered. This model explicitly incorporates the effect of nonuniform phase distributions of components through the von Mises PDF [9], which is a known PDF in communications [10]-[11]. By taking into account the randomness of the number of paths via the negative binomial distribution, the gamma PDF is derived for the envelope. So the main contribution of this paper is the introduction of gamma PDF for the envelope in multipath fading channels having random number of paths and nonuniform phase distributions.

2-A general model for multipath fading channels

Consider Q clusters of scatterers, which are distributed arbitrarily in space. In the jth cluster, there are N_j scatterers; and it is assumed that N_j is not too small. Each scatterer in the jth cluster reflects the incident wave with an attenuation factor L_j and a phase shift Θ_{js} ($j = 1,\ldots,Q$ and $s = 1,\ldots,N_j$). All Θ_{js}s are independent; and for a fixed j, all Θ_{js}s are distributed identically. The PDF of an individual Θ_{js}, when reduced to module 2π, is assumed to be of von Mises type with mean $\mu_{0,j}$ and concentration parameter κ_j [9]:

$$f_{\Theta_{js}}(\theta_{js}) = \frac{1}{2\pi I_0(\kappa_j)} \exp[\kappa_j \cos(\theta_{js} - \mu_{0,j})], \quad \kappa_j \ge 0,\ 0 \le \mu_{0,j} < 2\pi\,, \tag{1}$$

where $I_0(.)$ is the modified Bessel function of order zero. It should be mentioned that Q and all N_js and L_js can be either deterministic or random variables.

Among the available phase distributions [9], von Mises PDF has several attractive features. It plays a prominent role in statistical inference on the circle and its importance is almost the same as the Gaussian distribution on the line. This PDF usually results in mathematically tractable formulas. It can approximate other important phase PDFs quite well; and also contains two important PDFs as special cases: uniform on $[0,2\pi[$ for $\kappa = 0$ and impulse at $\mu_{0,j}$ for $\kappa = \infty$. Note that no restriction is imposed on the PDFs of Q, N_js and L_js, except N_js should not take small values (as becomes clear later).

3-Envelope PDF conditioned on Q , N_js and L_js

If the line-of-sight wave has unit amplitude and zero phase (a unit-length, zero-angle vector), then the reflected wave from the sth scatterer in the jth cluster can be considered as a vector with length L_j and angle Θ_{js}. By a simple extension of formula (4.6.5) in [9], the envelope PDF for the underlying channel model, conditioned on Q, N_js and L_js, can be expressed by the following exact formula:

$$f_R(r|Q,\boldsymbol{N},\boldsymbol{L}) = \frac{1}{2\pi\prod_{j=1}^{Q} I_0(\kappa_j)^{N_j}} \; r\int_{\rho=0}^{\infty} \int_{\Phi=0}^{2\pi} J_0(r\rho)\prod_{j=1}^{Q} J_0(W_j)^{N_j} \;\rho\; d\rho\; d\Phi,$$

$$0 \le r \le \sum_{j=1}^{Q} N_j L_j, \qquad (2)$$

where R is the univariate random variable of envelope, $J_0(.)$ is the Bessel function of order zero, $i = \sqrt{-1}$, and W_j is defined by:

$$W_j = [\rho^2 L_j^{\,2} - \kappa_j^{\,2} - 2\, i\, \rho\, L_j\, \kappa_j\, \cos(\Phi - \mu_{0,j})]^{1/2}. \qquad (3)$$

Moreover, $\boldsymbol{N}$ and $\boldsymbol{L}$ are Q-element vectors, with N_j and L_j as their jth elements.

4-The gamma envelope PDF

In order to obtain a simple and closed form PDF for the envelope, suppose Q is deterministic and let $Q = 1$. Moreover, let L_1 take the constant value ℓ. By a simple scaling, from (4.5.6) in [9] we get:

$$f_R(r|N) = \frac{1}{I_0(\kappa)^N} I_0\left(\frac{\kappa}{\ell} r\right) r\int_{\rho=0}^{\infty} \rho\, J_0(r\rho) J_0(\ell\rho)^N\, d\rho, \quad 0 \le r \le \ell N, \qquad (4)$$

where $\kappa = \kappa_1$ and $N = N_1$. Without loss of generality, suppose $\mu_{0,1} = 0$. Then if N is large enough [9], the mean and variance of R/N are given by $\ell\tau$ and $\ell^2\eta/N$, respectively, where $\tau = I_1(\kappa)/I_0(\kappa)$, $I_1(.)$ is the modified Bessel function of order one, and $\eta = [\kappa(1-\tau^2) - \tau]/\kappa$. As κ goes from 0 to ∞, η decreases from 0.5 to 0. Now we approximate R/N by a Gaussian random variable with the same mean and variance. A look at Fig. 1 reveals that this Gaussian approximation is reasonable when N is large and its accuracy improves as N increases. For a larger κ, smaller N is required to get the same approximation. If we define $\tilde{R} = R/\sqrt{\overline{N}}$, where $\overline{N}$ is the mean of N, then based on that Gaussian approximation, the characteristic function of $\tilde{R}$ conditioned on N, i.e. $\Psi_{\tilde{R}}(\omega|N) = E[\exp(i\omega\tilde{R})|N]$, can be written as:

$$\Psi_{\tilde{R}}(\omega|N) \approx \exp\left[iN\ell\tau\frac{\omega}{\sqrt{\overline{N}}} - \frac{N\ell^2\eta}{2}\left(\frac{\omega}{\sqrt{\overline{N}}}\right)^2\right] = \exp\left[(i\ell\sqrt{\overline{N}}\tau\omega - \frac{\ell^2\eta}{2}\omega^2)M\right]. \qquad (5)$$

In the above formula, the new random variable M is defined by $M = N/\overline{N}$.

Among the known discrete distributions, Poisson distribution is the most popular one with attractive analytic properties. It has already been used in modeling scattering phenomena where the number of scatterers is random [12]-[14]. However, the negative binomial distribution has become increasingly popular as a more flexible alternative to the Poisson distribution, specially when it is doubtful whether the strict requirements, particularly independence, for a Poisson distribution is satisfied [15]. In fact, Poisson distribution is the limiting form of the negative binomial distribution [15]. From another point of view, negative binomial distribution may be used to model variable-mean Poisson distribution [16]. Based on these evidences, we assume that *N* has a negative binomial distribution (It should be mentioned that the widely-used *K* PDF is also obtained assuming negative binomial distribution for the number of scatterers [17]-[19]):

$$f_N(n) = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \frac{(\overline{N}/\alpha)^n}{(1+\overline{N}/\alpha)^{n+\alpha}}, \tag{6}$$

where $\Gamma(.)$ is the gamma function and parameter α is a constant such that $\alpha > 0$. For large $\overline{N}$, the random variable *M* will have a gamma PDF [19]:

$$f_M(m) = \frac{\alpha^\alpha}{\Gamma(\alpha)} m^{\alpha-1} e^{-\alpha m}. \tag{7}$$

By taking the expectation of $\Psi_{\tilde{R}}(\omega|\overline{N}M)$ in (5) with respect to M using $f_M(m)$ in (7) and according to [20] we get:

$$\Psi_{\tilde{R}}(\omega) \approx \left(1 - i\frac{\ell\sqrt{\overline{N}}\tau}{\alpha}\omega + \frac{\ell^2\eta}{2\alpha}\omega^2\right)^{-\alpha}. \tag{8}$$

Now consider the limiting case in which $\ell \to 0$ and $\overline{N} \to \infty$ such that $\ell\sqrt{\overline{N}}$ remains constant (Similar limiting process has been used in [19] to derive the *K* PDF). Therefore (8) simplifies to:

$$\lim \Psi_{\tilde{R}}(\omega) = \left(1 - i\frac{\omega}{\beta}\right)^{-\alpha}, \tag{9}$$

where β is a constant such that $\beta = \alpha \big/ \ell\sqrt{\overline{N}}\tau$. Formula (9) is the characteristic function of the gamma random variable [21]. Hence the PDF of the measured signal envelope, $\Re$, can be expressed as:

$$f_\Re(r) = \frac{\beta^{\,\alpha}}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}, \quad \alpha > 0,\ \beta > 0. \tag{10}$$

5-Discussion and conclusion

Due to way we have built the channel model, the parameters α and β of the proposed envelope PDF in (10) have special physical meanings:

Interpretation of α: According to (7) we have $\alpha = 1/Var[M]$. So a large α indicates that the variations of the number of paths around $\overline{N}$ is small.

Interpretation of β: We have already observed that $\beta \propto 1/\tau$. Since τ increases as κ increases, we conclude that β is a decreasing function of κ. So for a fixed α, a small β represents a large κ, which means that the multipath components at the receiver are approximately coherent.

Although not shown here explicitly, nonuniformity of phase distributions transforms the in-phase and quadrature components of the received signal to two correlated Gaussian random variables with different means and variances, when *N* takes large values (Note that for the simple case of Rayleigh envelope PDF, in-phase and quadrature components are two independent Gaussian random variables with zero means and the same variances, when *N* takes large values). As mentioned earlier, the exact PDF of the signal envelope for this case is reported in (4.6-28) of [7], in the form of an infinite series whose terms contain Bessel function products (Of course for large *N*, (4) must tend to (4.6-28)). In this contribution we have accurately approximated (4.6-28) by a Gaussian PDF. However, it seems that (4.6-28) can be approximated more accurately by a PDF which is not as simple as the Gaussian PDF, but its complexity is still tolerable (This is under study).

References

[1] H. Hashemi, “The indoor radio propagation channel,” *Proc. IEEE*, vol. 81, pp. 943-968, 1993.

[2] K. D. Ward, “Application of the *K* distribution to radar clutter-A review,” in *Porch. IEICE Int. Symp. Noise and Clutter Rejection in Radars and Imaging Sensors*, Kyoto: Japan, 1989, pp. 15-20.

[3] A. Abdi and S. Nader-Esfahani, “Non-Rayleigh envelope PDF in multipath fading channels - General remarks and polynomial expansion,” in preparation.

[4] A. Abdi, H. Hashemi, and S. Nader-Esfahani, “On the PDF of the sum of random vectors,” submitted to *IEEE Trans. Commun.*, June 1997.

[5] A. Abdi and S. Nader-Esfahani, “A general PDF for the signal envelope in multipath fading channels using Laguerre polynomials,” in *Proc. IEEE Vehic. Technol. Conf.*, Atlanta, GA, 1996, pp. 1428-1432.

[6] A. Abdi and S. Nader-Esfahani, “An optimum Laguerre expansion for the envelope PDF of two sine waves in Gaussian noise,” in *Proc. IEEE Southeastcon Conf.*, Tampa, FL, 1996, pp. 160-163.

[7] P. Beckmann, *Probability in Communication Engineering*. New York: Harcourt, Brace & World, 1967.

[8] P. Beckmann and A. Spizzichino, *The Scattering of Electromagnetic Waves From Rough Surfaces*, 2nd ed., Boston, MA: Artech House, 1987.

[9] K. V. Mardia, *Statistics of Directional Data*. London: Academic, 1972.

[10] A. J. Viterbi, “Optimum detection and signal selection for partially coherent binary communication,” *IEEE Trans. Inform. Theory*, vol. 11, pp. 239-246, 1965.

[11] H. Leib and S. Pasupathy, “The phase of a vector perturbed by Gaussian noise and differentially coherent receivers", *IEEE Trans. Inform. Theory*, vol. 34, pp. 1491-1501, 1988.

[12] A. A. Giordano and F. Haber, “Modeling of atmospheric noise,” *Radio Sci.*, vol. 7, pp. 1011-1023, 1972.

[13] P. N. Pusey, D. W. Schaefer, and D. E. Koppel, “Single-interval statistics of light scattered by identical independent scatterers,” *J. Phys. A: Math., Nucl. Gen.*, vol. 7, pp. 530-540, 1974.

[14] D. Middleton, “Canonical non-Gaussian noise models: Their implications for measurements and for prediction of receiver performance,” *IEEE Trans. Electromagn. Compat.*, vol. 21, pp. 209-220, 1979.

[15] N. L. Johnson, S. Kotz, and A. W. Kemp, *Univariate Discrete Distributions*, 2nd ed., New York: Wiley, 1992.

[16] T. Azzarelli, “General class of non-Gaussian coherent clutter models,” *IEE Proc. Radar, Sonar, Navig.*, vol. 142, pp. 61-70, 1995.

[17] E. Jakeman and P. N. Pusey, “Significance of *K* distribution in scattering experiments,” *Phys. Rev. Lett.*, vol. 40, pp. 546-550, 1978.

[18] E. Jakeman, “On the statistics of *K*-distributed noise,” *J. Phys. A: Math. Gen.*, vol. 13, pp. 31-48, 1980.

[19] S. H. Yueh, J. A. Kong, J. K. Jao, R. T. Shin, H. A. Zebker, and T. Le Toan, “*K*-distribution and multi-frequency polarimetric terrain radar clutter,” *J. Electro. Waves Applic.*, vol. 5, pp. 1-15, 1991.

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed., A. Jeffrey, Ed., San Diego, CA: Academic, 1994.

[21] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed., Singapore: McGraw-Hill, 1991.