Sum of Correlated Gamma Variates with Application to the Performance of Wireless Systems over Nakagami Channels [1]

Mohamed-Slim Alouini
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: <alouini@ece.umn.edu>

Ali Abdi
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: <abdi@ece.umn.edu>

Mostafa Kaveh
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: <kaveh@ece.umn.edu>

Abstract — **We extend the Moschopoulos representation for the probability density function (PDF) of the sum of independent gamma variates to the PDF of the sum of correlated gamma variates and provide an exact PDF-based approach for the performance analysis of many wireless communication systems over not necessarily independent identically distributed (i.i.d.) Nakagami fading channels. The key feature of this representation is that it is in the from of a single gamma series which implies that all the manipulations that can be performed for the i.i.d. case can now also be done for the non-i.i.d case. Aside from putting under the same umbrella many of the past results obtained via characteristic function (CF)-based approaches, the proposed PDF-based approach also allows the derivation of additional performance measures which are harder to analyze via CF-based approaches.**

I. Introduction

Performance analysis of maximal-ratio combining (MRC) diversity scheme and co-channel interference in cellular mobile radio systems requires determination of the statistics of the sum (over the L diversity paths or the N_I cochannel interferers (CCI)) of the squared envelopes of Nakagami faded signals, or equivalently the sum of gamma variates. With this in mind, we extend the Moschopoulos representation [1] for the probability density function (PDF) of the sum of independent gamma variates to the PDF of the sum of correlated gamma variates and provide an exact PDF-based approach for the performance analysis of many wireless communication systems over not necessarily independent identically distributed (i.i.d.) Nakagami fading channels.

II. Sum of Gamma Variates

Let $\{X_n\}_{n=1}^N$ be a set of N correlated not necessarily identically distributed gamma variates with shape parameter α and scale parameter β_n, respectively, and let ρ_{ij} denote the correlation coefficient between X_i and X_j. It can be shown that the PDF of $Y = \sum_{n=1}^N X_n$ can be expressed in the following single gamma series form [2]

$$p_Y(y) = \prod_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{\delta_k \; y^{N\alpha+k-1} \; e^{-y/\lambda_1}}{\lambda_1^{N\alpha+k} \; \Gamma(N\alpha+k)} \; U(y), \qquad (1)$$

where $U(\cdot)$ is the unit step function, $\lambda_1 = \min_n\{\lambda_n\}$, $\{\lambda_n\}_{n=1}^N$ are the eigenvalues of the matrix $A = D\,C$, where D is the $N \times N$ diagonal matrix with the entries $\{\beta_n\}_{n=1}^N$ and C is the $N \times N$ positive definite matrix defined by

$$C = \begin{bmatrix} 1 & \sqrt{\rho_{12}} & \cdots & \sqrt{\rho_{1N}} \\ \sqrt{\rho_{21}} & 1 & \cdots & \sqrt{\rho_{2N}} \\ \cdot & \cdot & \cdot & \cdot \\ \sqrt{\rho_{1N}} & \sqrt{\rho_{2N}} & \cdots & 1 \end{bmatrix}_{N\times N}, \qquad (2)$$

and the coefficients δ_k can be obtained recursively as

$$\begin{cases} \delta_0 \quad = 1 \\ \delta_{k+1} = \frac{\alpha}{k+1} \sum_{i=1}^{k+1} \left[\sum_{j=1}^{N} \left(1 - \frac{\lambda_1}{\lambda_j}\right)^i \right] \delta_{k+1-i}, k = 0, 1, \cdots \end{cases} \qquad (3)$$

III. Application

Consider an MRC diversity receiver in which the L diversity paths go through correlated Nakagami fading channels. The total signal-to-noise ratio (SNR) at the combiner output $\gamma_t = \sum_{l=1}^L \gamma_l$, where γ_l is the instantaneous SNR of the lth path. Hence, to obtain the PDF of γ_t we need to use the result of the previous section with the substitutions in (1) and (3) of N by L, y by γ_t, α by m, and β_l by $\overline{\gamma}_l/m$. This result gives an exact gamma-series representation to the approximate gamma solution presented in [3] and provides the starting point to unify many of the performance measures of MRC over Nakagami fading channels. For example, the zero-outage capacity per unit bandwidth [4] can be shown to be given by [2]

$$\frac{C_{\text{zout}}}{W} = \log_2 \left[1 + \frac{1}{\prod_{l=1}^{L} \lambda_l^{-m} \lambda_1^{Lm-1} \sum_{k=0}^{\infty} \frac{\delta_k}{Lm+k-1}} \right]. \qquad (4)$$

Various other performance measures of MRC and cellular mobile radio systems with CCI can also be treated as shown in [2].

References

[1] P. G. Moschopoulos, "The distribution of the sum of independent gamma random variables," *Ann. Inst. Statist. Math. (Part A)*, vol. 37, pp. 541–544, 1985.

[2] M. -S. Alouini, A. Abdi, and M. Kaveh, "Sum of gamma variates and performance of wireless communication systems over Nakagami fading channels," To appear in *IEEE Trans. Veh. Technol.*

[3] C. Mun, C. -H. Kang, and H. -K. Park, "Approximation of SNR statistics for MRC diversity in arbitrarily correlated Nakagami fading channels," *IEE Electron. Lett.*, vol. 35, pp. 266–267, February 1999.

[4] G. Caire and S. Shamai, "On the capacity of some channels with channel state information," *IEEE Trans. Inform. Theory*, vol. IT-45, pp. 2007–2019, September 1999.

[1] This work was supported by the National Science Foundation under the Wireless Initiative Grant Number 9979443.