Sum of Correlated Gamma Variates with Application to the Performance of Wireless Systems over Nakagami Channels

Mohamed-Slim Alouini
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: alouini@ece.umn.edu

Ali Abdi
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: abdi@ece.umn.edu

Mostafa Kaveh
Department of ECE
University of Minnesota
Minneapolis, MN 55455, USA
E-mail: kaveh@ece.umn.edu

Abstract — We extend the Moschopoulos representation for the probability density function (PDF) of the sum of independent gamma variates to the PDF of the sum of correlated gamma variates and provide an exact PDF-based approach for the performance analysis of many wireless communication systems over not necessarily independent identically distributed (i.i.d.) Nakagami fading channels. The key feature of this representation is that it is in the from of a single gamma series which implies that all the manipulations that can be performed for the i.i.d. case can now also be done for the non-i.i.d case. Aside from putting under the same umbrella many of the past results obtained via characteristic function (CF)-based approaches, the proposed PDF-based approach also allows the derivation of additional performance measures which are harder to analyze via CF-based approaches.

I. INTRODUCTION

Performance analysis of maximal-ratio combining (MRC) diversity scheme and co-channel interference in cellular mobile radio systems requires determination of the statistics of the sum (over the L diversity paths or the N_c cochannel interferers (CCI)) of the squared envelopes of Nakagami fading signals, or equivalently the sum of gamma variates. With this in mind, we extend the Moschopoulos representation [1] for the probability density function (PDF) of independent gamma variates to the PDF of the sum of correlated gamma variates and provide an exact PDF-based approach for the performance analysis of many wireless communication systems over not necessarily independent identically distributed (i.i.d.) Nakagami fading channels.

II. SUM OF GAMMA VARIATES

Let \{X_n\}_{n=1}^{N} be a set of N correlated not necessarily identically distributed gamma variates with shape parameter \(a_n\) and scale parameter \(\beta_n\), respectively, and let \(\rho_{ij}\) denote the correlation coefficient between \(X_i\) and \(X_j\). It can be shown that the PDF of \(Y = \sum_{n=1}^{N} X_n\) can be expressed in the following single gamma series form [2]

\[
p_Y(y) = \sum_{n=1}^{N} \left( \frac{\lambda_n}{\lambda} \right)^{a} \delta_n k^{N_k \alpha + k - 1} e^{-y/\lambda_k} U(y),
\]

where \(U(\cdot)\) is the unit step function, \(\lambda_k = \min\{\lambda_n\}\), \(\lambda_k\) are the eigenvalues of the matrix \(A = D C\), where \(D\) is the \(N \times N\) diagonal matrix with the entries \(\{\beta_n\}_{n=1}^{N}\) and \(C\) is the \(N \times N\) positive definite matrix defined by

\[
C = \begin{bmatrix}
\frac{1}{\beta_1^{\alpha}} & \frac{1}{\beta_2^{\alpha}} & \cdots & \frac{1}{\beta_N^{\alpha}} \\
\frac{1}{\beta_1^{\alpha}} & 1 & \cdots & \frac{1}{\beta_N^{\alpha}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{\beta_1^{\alpha}} & \frac{1}{\beta_2^{\alpha}} & \cdots & 1
\end{bmatrix}_{N \times N}, (2)
\]

and the coefficients \(\delta_k\) can be obtained recursively as

\[
\delta_0 = 1 \\
\delta_{k+1} = \frac{\alpha}{\lambda_k} \sum_{j=1}^{N_k} \left[ \frac{\lambda_k}{\lambda_j} \right]^{\alpha} \delta_k, k = 0, 1, \ldots
\]

III. APPLICATION

Consider an MRC diversity receiver in which the L diversity paths go through correlated Nakagami fading channels. The total signal-to-noise ratio (SNR) at the combiner output \(\gamma_t = \sum_{l=1}^{L} \gamma_l\), where \(\gamma_l\) is the instantaneous SNR of the \(l\)th path. Hence, to obtain the PDF of \(\gamma_t\) we need to use the result of the previous section with the substitutions in (1) and (3) of \(N\) by \(L\), \(y\) by \(\gamma\), \(\alpha\) by \(m\), and \(\beta\) by \(\gamma_l / m\). This result gives an exact gamma-series representation of the approximate gamma solution presented in [3] and provides the starting point to unify many of the performance measures of MRC over Nakagami fading channels. For example, the zero-outage capacity per unit bandwidth [4] can be shown to be given by [2]

\[
C_{\text{out}} = \log_2 \left[ 1 + \frac{1}{\prod_{l=1}^{L} \lambda_l^{-m} \lambda_l^{m-1} \sum_{k=0}^{m-1} \frac{2^k}{k^m}} \right]. \quad (4)
\]

Various other performance measures of MRC and cellular mobile radio systems with CCI can also be treated as shown in [2].

REFERENCES


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