ON THE UTILITY OF GAMMA PDF IN MODELING SHADOW FADING (SLOW FADING)

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Abstract - Shadow fading causes random fluctuations of the envelope mean at the mobile. The lognormal distribution is widely accepted for this phenomenon. In this paper we argue that, based on theoretical results and measured data, gamma distribution does the job as well. Then we show that by using the gamma distribution and in contrast with the use of the lognormal distribution, we can obtain easy-to-use and closed-form composite PDFs for fading channels, which in turn extremely simplify analytic calculations.

I. INTRODUCTION

The mean of the signal envelope at the mobile station (MS), the envelope averaged over a distance of a few wavelengths, is a random variable due to shadow variations generated by large terrain characteristics between the base station (BS) and MS, like buildings and hills in macrocells and vehicles in microcells. This phenomenon is generally referred to as shadow fading (slow fading). The probability density function (PDF) of the envelope mean is widely accepted to be lognormal [1, p. 87] [2, p. 59] [3] [4, p. 49] [5, p. 8] [6, p. 88] [Par 92, p. 151] [8, p. 163]. While for multipath fading (fast fading) numerous PDFs like Rayleigh, Rice, Nakagami, Weibull, ... have been examined for describing the data, it appears that there are no comparative theoretical and/or experimental studies on alternatives to the lognormal PDF for modeling shadow fading. In this paper we show that based on theoretical and experimental evidences, the gamma PDF is an excellent candidate for describing shadow fading and discuss numerous advantages resulting from this PDF.

II. THEORETICAL FACTS

The main justification for relating the lognormal PDF to the envelope mean fluctuations is the multiplicative model [3] [4, p. 51] [6, p. 89]. The applicability of this model is quantified in [9] through Monte Carlo simulations, which can also be done using analytic tools [10]. This model has been related to the physical characteristics of the channel as well [11].

On the other hand, theoretical arguments based on scattering from random rough surfaces with fractal slopes give rise to the gamma PDF for the local mean [12] [13].

The cumulative distribution function (CDF) of a lognormal random variable, when plotted in terms of logarithmic values, appears as a straight line [5, p. 29]. Because of the ease of fitting straight lines to experimental data, this common and ad-hoc graphical method has often been used to check the validity of the lognormal assumption for a given data set. For the gamma random variable, however, we get slightly to moderately convex curves instead of straight lines [14] [15]. A look at graphical representation of the empirical CDFs at MS versus the signal level in dB [4, p. 50] [7, p. 152] reveals that curved lines fit at least as well as the straight lines to the mobile communication channel data, and in some cases fit better. This suggests the application of the gamma PDF for the mean of the received envelope at the MS. The same issue has been discussed, although generally overlooked, for the distribution of the received power in VHF and UHF tropospheric transmission [15]. Note that the above qualitative discussion has strong mathematical support. In fact, gamma and lognormal can closely approximate each other [16, p. 196] [17] [18].

III. EXPERIMENTAL RESULTS

Narrowband inphase and quadrature component data have been collected at f =910.25 MHz UHF carrier in two different suburban areas [19]. We have selected twelve envelope records form that database. Each envelope sample has been computed according to $r=(i^2+q^2)^{1/2}$, where r is the envelope and i and q represent inphase and quadrature components. In the upper part of Figs. 1-4, four envelope records are shown with the envelope means $m=E[r]$ superimposed, while the envelopes divided by the envelope means, i.e. r/m, are plotted separately. Note that E denotes expectation.

The two important issues in calculating the envelope local mean, i.e. the length of the window over which the averaging should be done and the number of samples within the window, are discussed in [1, pp. 509-514] [4, pp. 114-117] [5, pp. 47-51] [7, pp. 214-220] [8, pp. 170-173] [20, pp. 367-371] [21, pp. 169-171]. Each point of the envelope mean has been calculated by averaging the envelope samples over a sliding window of typical length [1, p. 514] $20\lambda=20\ c/f=20(300/910.25)=6.6$ m, where λ is the carrier wavelength and c is the speed of light. The speed of the mobile receiver was fixed at 6.7 m/s, so the length of the sliding window is approximately 1 s. The sampling frequency was 35156.25 Hz, so any two successive samples are separated by 28 μs, or 0.2 mm=$0.57\times10^{-3}\lambda<0.5\lambda$ [1, p. 514].

The lognormal PDF has the form:

$$f_L(m)=\frac{1}{\sqrt{2\pi\sigma^2}\,m}\exp\left[\frac{-(\ln m-\mu)^2}{2\sigma^2}\right], \qquad (1)$$

with $E[m]=\exp(\mu+\sigma^2/2)$ and $E[m^2]=\exp(2\mu+2\sigma^2)$. For estimating the parameters μ and σ, natural logarithm of m, i.e. ln m, has been considered and μ and σ have been estimated according to:

$$\mu=E[\ln m],\ \ \sigma=\sqrt{V[\ln m]}\,, \qquad (2)$$

where V means variance. The gamma PDF has the form:

$$f_G(m)=\frac{1}{b^a\Gamma(a)}m^{a-1}e^{-m/b}\,, \qquad (3)$$

where $\Gamma(.)$ is the gamma function. In this case $E[m]=ab$ and $E[m^2]=a(a+1)b^2$. Therefore the parameters a and b have been estimated directly from m based on the following formulas:

$$a=\frac{E[m]^2}{V[m]},\ \ b=\frac{V[m]}{E[m]}. \qquad (4)$$

By equating the first and the second moments of lognormal and gamma PDFs, we obtain the relationships between μ, σ, a, and b:

$$a=[\exp(\sigma^2)-1]^{-1}, b=\exp(\mu+\sigma^2/2)[\exp(\sigma^2)-1]\,, \qquad (5)$$

$$\mu=\ln[ab(1+1/a)^{-1/2}], \sigma=\sqrt{\ln(1+1/a)}\,. \qquad (6)$$

Note that based on (5), a is a monotonically decreasing function of σ.

In Table I, estimated values of μ, σ, a, and b are listed for all records. The dB values are computed based on 20 log m=(20/ln10)ln m=8.68 ln m. So $\sigma_{dB}=8.68\ \sigma$. Based on Table I, lognormal and gamma CDFs of 0011, 0013, 0018, and 0021 are shown in the lower part of Figs. 1-4, together with the empirical CDFs.

As is expected from the theory [16, p. 196], when a is not too small (σ is not too large), lognormal and gamma CDFs are close (lower part of Figs. 1, 2, 3). We can also see good fit of both lognormal and gamma CDFs to the empirical CDFs in those figures. The value of a (σ) for the record 0021 is small (large), so as can be seen in the lower part of Fig. 4, the CDFs of lognormal and gamma are not close. Moreover, none of them is superior to the other one in terms of describing the empirical CDF. Based on these observations (and also those that are not reported in this paper), we can conclude that for real data, gamma PDF can describe shadow fading as well as the lognormal PDF.

IV. APPLICATIONS OF GAMMA PDF

Although lognormal and gamma PDFs are similar in modeling real data, gamma is much superior to the lognormal for analytic computations that arise in connection with shadow fading. In fact, the mathematical form of the lognormal PDF is such that we cannot get closed-form and easy-to-use formulas for many system calculations like average bit error rate (BER) computations, effect of various diversity schemes on average BERs, outage probability calculations in the presence of multiple interferers, coverage prediction, etc. in wireless channels with shadow fading. In what follows, we show how clean and closed-form PDFs can be obtained for multipath-shadow fading channels, upon application of the gamma PDF for shadow fading. Using these mathematically tractable PDFs, the above mentioned system calculations can be easily done in closed forms. It should be noted that in the literature, lognormal PDF has been assigned to various quantities other than $E[r]$,

TABLE I

ESTIMATED VALUES OF μ, σ, a, AND b

record #	μ	σ (σ_{dB})	a	b
0011	7.04	0.540 (4.7)	3.06	432
0012	6.89	0.321 (2.8)	8.24	126
0013	6.05	0.393 (3.4)	8.38	54.3
0014	7.43	0.353 (3.1)	7.92	226
0015	5.06	0.259 (2.2)	15.3	10.6
0016	6.31	0.282 (2.4)	13.3	43.1
0017	6.15	0.405 (3.5)	6.25	81.9
0018	5.86	0.196 (1.7)	28.1	12.7
0019	4.12	0.220 (1.9)	31.1	2.01
0020	5.47	0.196 (1.7)	23.7	10.2
0021	5.54	0.815 (7.1)	1.09	336
0022	6.01	0.187 (1.6)	40.7	10.2

e.g. $E[r^2]/2$, $(2/\pi)^{1/2}E[r]$, and $E[r^2]$ [22]. So the gamma PDF can be applied to all of them as well.

Substitute for Rayleigh-lognormal PDF: The Rayleigh PDF with lognormally-distributed power, introduced by Hansen and Meno [23] (widely known as the Suzuki PDF), is popular in modeling multipath-shadow fading channels. Because of the complicated integral form of this PDF, it is impossible to obtain closed-form results for various system calculations. Even fitting this PDF to real data is a difficult task. Now let us replace the lognormal with gamma. Then the Rayleigh PDF with gamma-distributed power yields the *K* PDF [23], widely used in radar and optics literature for describing random scattering from rough surfaces. Because of the extensive application of the *K* PDF in those areas, many results are available on the simulation and parameter estimation for the *K* PDF. As an example of the benefits obtained by replacing Rayleigh-lognormal with *K*, we have shown that the average BER of BPSK and BDPSK can be expressed cleanly in terms of known mathematical functions [22]. Even for the simple BER expression of BDPSK, average BER in the presence of Rayleigh-lognormal PDF takes a very complicated form [24]. Similarly, closed-form results can be obtained for outage probability and diversity computations, specially because the characteristic function and moment generating function of *K* PDF have convenient mathematical forms.

Substitute for Nakagami-lognormal PDF: The Nakagami PDF with lognormally-distributed power [1, p.92] is very useful for modeling multipath-shadow fading, because the flexible Nakagami PDF which is mathematically convenient, includes Rayleigh PDF as a special case, closely approximates Rice PDF, etc. The Nakagami-lognormal PDF has a more complicated integral form than the Rayleigh-lognormal PDF. Interestingly, the Nakagami PDF with gamma-distributed power simplifies to a three-parameter *K* PDF (the previous *K* PDF has two parameters and in fact is a special case of this three-parameter *K* PDF, because Rayleigh is a special case of Nakagami). Hence, many link performance evaluations for this very general composite PDF can be easily done in terms of tabulated mathematical functions.

Substitute for lognormal-lognormal, lognormal-gamma, and gamma-lognormal PDFs: Moulsley and Vilar's lognormal-lognormal PDF [25] and Karasawa's lognormal-gamma PDF [26], both useful for modeling signal amplitude scintillation over different satellite links [27] [28] can be replaced by the gamma-gamma PDF. It is shown that the gamma-gamma PDF is nothing but the easy-to-use three-parameter *K* PDF [29]. Note that both the lognormal-lognormal and lognormal-gamma PDFs have complicated integral forms, and only for the former an approximation is proposed [30], which still is not suitable for analytic manipulations, contrary to the *K* PDF. The gamma-lognormal PDF is nothing but the Nakagami squared-lognormal PDF [1, p.92].

V. CONCLUSION

The lognormal PDF has been widely accepted for shadow fading, with only few alternatives proposed against it, like the Nakagami PDF [31]. In this paper we have shown that based on theoretical facts and experimental results, shadow fading can be modeled by the gamma PDF as well as the lognormal PDF. In this way, for fading channels, we obtain closed-form PDFs which significantly simplify system analysis and design issues. The modified Bessel function of the second kind and order ν, $K_\nu(.)$, which is the core of those PDFs, also appears in other propagation-related PDFs [32] [33].

In addition to the inconvenient mathematical form of lognormal, this PDF cannot be identified uniquely from its moments [34] [35]. This fact may or may not affect the utility of lognormal PDF.

Lognormal PDF has been found suitable for multipath fading (i.e., fast fading over local areas) in indoor channels [36]. According to the approximate equivalence of gamma and lognormal, we may use the gamma PDF for the same purpose. A theoretical explanation of the gamma PDF for the envelope in propagation environments with random number of paths and nonuniform phase distributions is given in [37].

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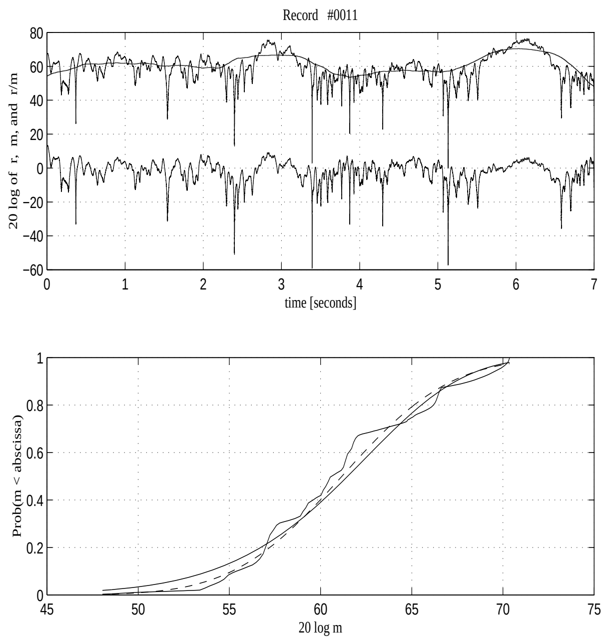
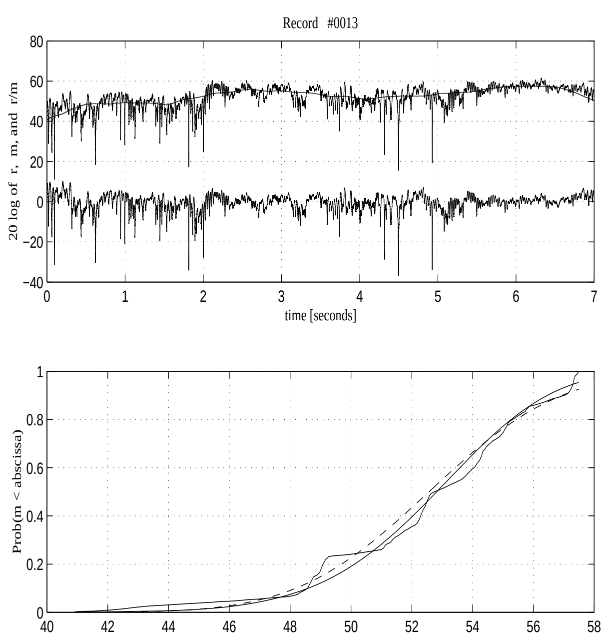


Fig. 1. Upper− envelope *r*, envelope mean *m*, and *r*/*m*, Lower− empirical & gamma (solid), lognormal (dashed)



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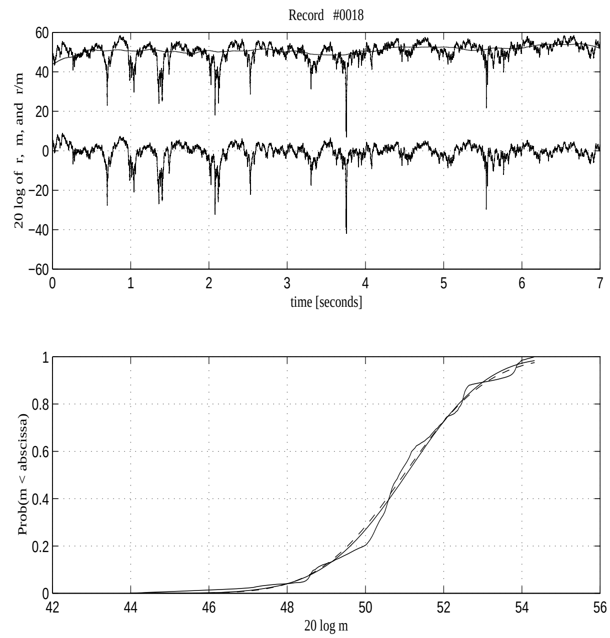

Fig. 3. Upper− envelope *r*, envelope mean *m*, and *r*/*m*, Lower− empirical & gamma (solid), lognormal (dashed)

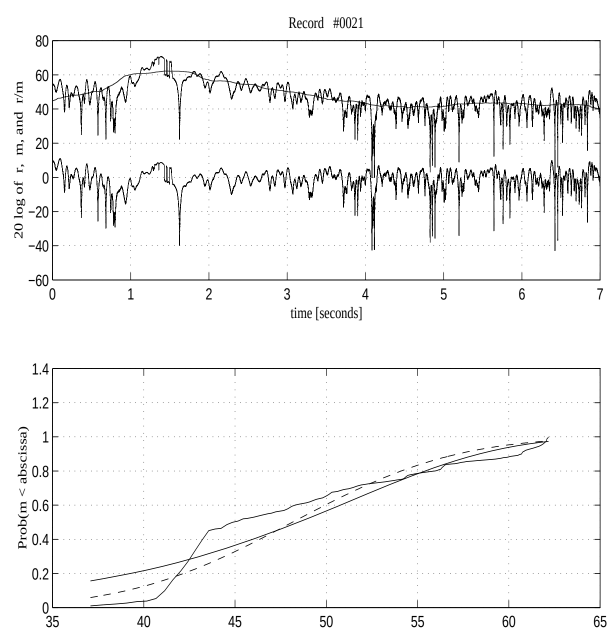

Fig. 4. Upper− envelope *r*, envelope mean *m*, and *r*/*m*, Lower− empirical & gamma (solid), lognormal (dashed)