Digital Processing of Continuous-Time Signals

- Digital processing of a continuous-time signal involves the following basic steps:
  1. Conversion of the continuous-time signal into a discrete-time signal,
  2. Processing of the discrete-time signal,
  3. Conversion of the processed discrete-time signal back into a continuous-time signal

Conversion of a continuous-time signal into digital form is carried out by an analog-to-digital (A/D) converter.

The reverse operation of converting a digital signal into a continuous-time signal is performed by a digital-to-analog (D/A) converter.

Since the A/D conversion takes a finite amount of time, a sample-and-hold (S/H) circuit is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation.

To prevent aliasing, an analog anti-aliasing filter is employed before the S/H circuit.

To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog reconstruction filter is used.

Complete block-diagram

- Anti-aliasing filter
- S/H
- A/D
- Digital processor
- D/A
- Reconstruction filter

Sampling of Continuous-Time Signals

- As indicated earlier, discrete-time signals in many applications are generated by sampling continuous-time signals.
- We have seen earlier that identical discrete-time signals may result from the sampling of more than one distinct continuous-time function.

Since both the anti-aliasing filter and the reconstruction filter are analog lowpass filters, we review first the theory behind the design of such filters.

Also, the most widely used IIR digital filter design method is based on the conversion of an analog lowpass prototype.
Sampling of Continuous-Time Signals

- In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal.
- However, under certain conditions, it is possible to relate a unique continuous-time signal to a given discrete-time signal.

Effect of Sampling in the Frequency Domain

- Let \( g_a(t) \) be a continuous-time signal that is sampled uniformly at \( t = nT \), generating the sequence \( g[n] \) where \( g[n] = g_a(nT), \quad -\infty < n < \infty \) with \( T \) being the sampling period.
- The reciprocal of \( T \) is called the sampling frequency \( F_T \), i.e., \( F_T = \frac{1}{T} \).

To establish the relation between \( G_a(j\Omega) \) and \( G(e^{j\omega}) \), we treat the sampling operation mathematically as a multiplication of \( g_a(t) \) by a periodic impulse train \( p(t) \):

\[
p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]

\[
g_a(t) \quad p(t)
\]

\[
g_p(t) = g_a(t)p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)
\]

Effect of Sampling in the Frequency Domain

- Now, the frequency-domain representation of \( g_a(t) \) is given by its continuous-time Fourier transform (CTFT):

\[
G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t)e^{-j\Omega t} dt
\]

- The frequency-domain representation of \( g[n] \) is given by its discrete-time Fourier transform (DTFT):

\[
G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}
\]
Effect of Sampling in the Frequency Domain

• $g_P(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant.

Effect of Sampling in the Frequency Domain

• The impulse train $g_p(t)$ therefore can be expressed as:
  
  $$g_p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\Omega_k t} \cdot g_a(t)$$

• From the frequency-shifting property of the CTFT, the CTFT of $e^{j\Omega_k t} g_a(t)$ is given by:
  $G_a(j(\Omega - k\Omega_T))$

Effect of Sampling in the Frequency Domain

• The term on the RHS of the previous equation for $k = 0$ is the baseband portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$.

• The frequency range $-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$

• is called the baseband or Nyquist band.

Effect of Sampling in the Frequency Domain

• There are two different forms of $G_p(j\Omega)$:
  
  • One form is given by the weighted sum of the CTFTs of $\delta(t - nT)$:
    $$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j2\pi nT \Omega}$$
  
  • To derive the second form, we note that $p(t)$ can be expressed as a Fourier series:
    $$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j(2\pi/T)kT} = \sum_{k=-\infty}^{\infty} e^{j\Omega_T k}$$
    where $\Omega_T = 2\pi/T$

Effect of Sampling in the Frequency Domain

• Hence, an alternative form of the CTFT of $g_p(t)$ is given by:
  $G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$

• Therefore, $G_p(j\Omega)$ is a periodic function of $\Omega$ consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of $\Omega_T$ and scaled by $\frac{1}{T}$.

Effect of Sampling in the Frequency Domain

• Assume $g_a(t)$ is a band-limited signal with a CTFT $G_a(j\Omega)$ as shown below.

• The spectrum $P(j\Omega)$ of $p(t)$ having a sampling period $T = \frac{2\pi}{\Omega_T}$ is indicated below.
Effect of Sampling in the Frequency Domain

- Two possible spectra of $G_p(j\Omega)$ are shown below.

\[
\begin{align*}
G_p(j\Omega) & = G_p(j\Omega_1), \\
G_p(j\Omega) & = G_p(j\Omega_2).
\end{align*}
\]

It is evident from the top figure on the previous slide that if $\Omega_T > 2\Omega_m$, there is no overlap between the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$.

- On the other hand, as indicated by the figure on the bottom, if $\Omega_T < 2\Omega_m$, there is an overlap of the spectra of the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$.

If $\Omega_T > 2\Omega_m$, $g_a(t)$ can be recovered exactly from $g_a(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain $T$ and a cutoff frequency $\Omega_c$ greater than $\Omega_m$ and less than $\Omega_T - \Omega_m$ as shown below.

\[
\begin{align*}
g_a(t) \rightarrow H_r(j\Omega) \rightarrow g_a(t)
\end{align*}
\]

On the other hand, if $\Omega_T < 2\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_a(j\Omega)$ cannot be separated by filtering to recover $G_a(j\Omega)$ because of the distortion caused by a part of the replicas immediately outside the baseband folded back or aliased into the baseband.

Effect of Sampling in the Frequency Domain

- The spectra of the filter and pertinent signals are shown below.

- Sampling theorem: Let $g_a(t)$ be a band-limited signal with CTFT $G_a(j\Omega) = 0$ for $|\Omega| > \Omega_m$.

- Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty < n \leq \infty$ if

$$
\Omega_T \geq 2\Omega_m
$$

where $\Omega_T = 2\pi/T$. 
Effect of Sampling in the Frequency Domain

- The condition $\Omega_T \geq 2\Omega_m$ is often referred to as the Nyquist condition.
- The frequency $\frac{\Omega_T}{2}$ is usually referred to as the folding frequency.

Effect of Sampling in the Frequency Domain

- Given $g_a(nT)$, we can recover exactly $g_a(t)$ by generating an impulse train $g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$ and then passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain $T$ and a cutoff frequency $\Omega_c$ satisfying $\Omega_m < \Omega_c < (\Omega_T - \Omega_m)$.

Effect of Sampling in the Frequency Domain

- The highest frequency $\Omega_m$ contained in $g_a(t)$ is usually called the Nyquist frequency since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$ that must be used to fully recover $g_a(t)$ from its sampled version.
- The frequency $2\Omega_m$ is called the Nyquist rate.

Effect of Sampling in the Frequency Domain

- Oversampling - The sampling frequency is higher than the Nyquist rate.
- Undersampling - The sampling frequency is lower than the Nyquist rate.
- Critical sampling - The sampling frequency is equal to the Nyquist rate.
- Note: A pure sinusoid may not be recoverable from its critically sampled version.

Effect of Sampling in the Frequency Domain

- In digital telephony, a 3.4 kHz signal bandwidth is acceptable for telephone conversation.
- Here, a sampling rate of 8 kHz, which is greater than twice the signal bandwidth, is used.

Effect of Sampling in the Frequency Domain

- In high-quality analog music signal processing, a bandwidth of 20 kHz has been determined to preserve the fidelity.
- Hence, in compact disc (CD) music systems, a sampling rate of 44.1 kHz, which is slightly higher than twice the signal bandwidth, is used.
Effect of Sampling in the Frequency Domain

**Example** - Consider the three continuous-time sinusoidal signals:

\[ g_1(t) = \cos(6\pi t) \]
\[ g_2(t) = \cos(14\pi t) \]
\[ g_3(t) = \cos(26\pi t) \]

Their corresponding CTFTs are:

\[ G_1(j\Omega) = \pi[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)] \]
\[ G_2(j\Omega) = \pi[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)] \]
\[ G_3(j\Omega) = \pi[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)] \]

These continuous-time signals sampled at a rate of \( T = 0.1 \) sec, i.e., with a sampling frequency \( \Omega_T = 20\pi \) rad/sec

The sampling process generates the continuous-time impulse trains, \( g_{1p}(t) \), \( g_{2p}(t) \), and \( g_{3p}(t) \)

Their corresponding CTFTs are given by

\[ G_{lp}(j\Omega) = 10\sum_{k=-\infty}^{\infty} G_l(j(\Omega - k\Omega_T)), \quad 1 \leq l \leq 3 \]

These figures also indicate by dotted lines the frequency response of an ideal lowpass filter with a cutoff at \( \Omega_c = \Omega_T / 2 = 10\pi \) and a gain \( T = 0.1 \)

The CTFTs of the lowpass filter output are also shown in these three figures

In the case of \( g_1(t) \), the sampling rate satisfies the Nyquist condition, hence no aliasing

Moreover, the reconstructed output is precisely the original continuous-time signal

In the other two cases, the sampling rate does not satisfy the Nyquist condition, resulting in aliasing and the filter outputs are all equal to \( \cos(6\pi t) \)
Effect of Sampling in the Frequency Domain

- Note: In the figure below, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_2(j\Omega)$ at $\Omega = -14\pi$
- Likewise, the impulse appearing at $\Omega = 6\pi$ in the positive frequency passband of the filter results from the aliasing of the impulse in $G_3(j\Omega)$ at $\Omega = 26\pi$

Effect of Sampling in the Frequency Domain

- Observation: We have
  
  $$G(e^{j\omega}) = G_p(j\Omega) \bigg|_{\Omega = \omega/T}$$
  
  or, equivalently,
  
  $$G_p(j\Omega) = G(e^{j\omega}) \bigg|_{\omega = \Omega T}$$
  
  From the above observation and
  
  $$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

Effect of Sampling in the Frequency Domain

- The relation derived on the previous slide can be alternately expressed as
  
  $$G(e^{j\omega}) = G_p(j\Omega) \bigg|_{\Omega = \omega/T}$$
  
  or from
  
  $$G_p(j\Omega) = G(e^{j\omega}) \bigg|_{\omega = \Omega T}$$
  
  it follows that $G(e^{j\omega})$ is obtained from $G_p(j\Omega)$ by applying the mapping $\Omega = \frac{\omega}{T}$

Effect of Sampling in the Frequency Domain

- We now derive the relation between the DTFT of $g[n]$ and the CTFT of $g_p(t)$
- To this end we compare
  
  $$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$
  
  with
  
  $$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$
  
  and make use of $g[n] = g_a(nT)$, $-\infty < n < \infty$

Effect of Sampling in the Frequency Domain

- Now, the CTFT $G_p(j\Omega)$ is a periodic function of $\Omega$ with a period $\Omega_T = 2\pi / T$
- Because of the mapping, the DTFT $G(e^{j\omega})$ is a periodic function of $\omega$ with a period $2\pi$
Recovery of the Analog Signal

• We now derive the expression for the output \( \hat{g}_a(t) \) of the ideal lowpass reconstruction filter \( H_r(j\Omega) \) as a function of the samples \( g[n] \).
• The impulse response \( h_r(t) \) of the lowpass reconstruction filter is obtained by taking the inverse DTFT of \( \widehat{H_r(j\Omega)} \):
\[
H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}
\]

• Thus, the impulse response is given by:
\[
h_r(t) = \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT)
\]
• The input to the lowpass filter is the impulse train \( g_p(t) \):
\[
g_p(t) = \sum_{n=-\infty}^{\infty} g[n] \delta(t - nT)
\]

Recovery of the Analog Signal

• Therefore, the output \( \hat{g}_a(t) \) of the ideal lowpass filter is given by:
\[
\hat{g}_a(t) = h_r(t) \circledast g_a(t) = \sum_{n=-\infty}^{\infty} g[n] h_r(t - nT)
\]

• Substituting \( h_r(t) = \sin(\Omega_c t) / (\Omega_c T / 2) \) in the above and assuming for simplicity \( \Omega_c = \Omega_c / 2 = \pi / T \), we get:
\[
\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin(\pi(t - nT) / T)}{\pi(t - nT) / T}
\]

Recovery of the Analog Signal

• It can be shown that when \( \Omega_c = \Omega_c / 2 \) in
\[
h_r(t) = \frac{\sin(\Omega_c t)}{\Omega_c T / 2}
\]
h\(_r(0) = 1 \) and \( h_r(nT) = 0 \) for \( n \neq 0 \).
• As a result, from
\[
\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin(\pi(t - nT) / T)}{\pi(t - nT) / T}
\]
we observe
\[
g_a(rT) = g[r] = \hat{g}_a(rT)
\]
for all integer values of \( r \) in the range \( -\infty < r < \infty \).

Recovery of the Analog Signal

• The relation
\[
\hat{g}_a(rT) = g[r] = g_a(rT)
\]
holds whether or not the condition of the sampling theorem is satisfied.
• However, \( \hat{g}_a(rT) = g_a(rT) \) for all values of \( r \) only if the sampling frequency \( \Omega_c \) satisfies the condition of the sampling theorem.
Implication of the Sampling Process

• Consider again the three continuous-time signals: $g_1(t) = \cos(6\pi t)$, $g_2(t) = \cos(14\pi t)$, and $g_3(t) = \cos(26\pi t)$
• The plot of the CTFT $G_{\Omega_p}(j\Omega)$ of the sampled version $g_{\Omega_p}(t)$ of $g_1(t)$ is shown below.

• From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos[(20k \pm 6)\pi t]$ outside the baseband by passing $g_{\Omega_p}(t)$ through an ideal analog bandpass filter with a passband centered at $\Omega = (20k \pm 6)\pi$

• For example, to recover the signal $\cos(34\pi t)$, it will be necessary to employ a bandpass filter with a frequency response $H_r(j\Omega) = \begin{cases} 0, & (34 - \Delta)\pi \leq \Omega \leq (34 + \Delta)\pi \\ 0, & \text{otherwise} \end{cases}$ where $\Delta$ is a small number.

• Likewise, we can recover the aliased baseband component $\cos(6\pi t)$ from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response:
$H_r(j\Omega) = \begin{cases} 0, & (6 - \Delta)\pi \leq \Omega \leq (6 + \Delta)\pi \\ 0, & \text{otherwise} \end{cases}$

• There is no aliasing distortion unless the original continuous-time signal also contains the component $\cos(6\pi t)$
• Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $g_2(t)$ or $g_3(t)$ as the case may be, by employing suitable filters.

Sampling of Bandpass Signals

• The conditions developed earlier for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the continuous-time signal is bandlimited in the frequency range from dc to some frequency $\Omega_m$
• Such a continuous-time signal is commonly referred to as a lowpass signal.
Sampling of Bandpass Signals

- There are applications where the continuous-time signal is bandlimited to a higher frequency range $\Omega_L \leq \Omega \leq \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the **bandpass signal**
- To prevent aliasing a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring $\Omega_T \geq 2\Omega_H$

Sampling of Bandpass Signals

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if $\Omega_H$ is very large, the sampling rate also has to be very large which may not be practical in some situations

Sampling of Bandpass Signals

- A more practical approach is to use undersampling
- Let $\Delta \Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency $\Omega_H$ contained in the signal is an integer multiple of the bandwidth, i.e., $\Omega_H = M(\Delta \Omega)$

Sampling of Bandpass Signals

- We choose the sampling frequency $\Omega_T$ to satisfy the condition $\Omega_T = 2(\Delta \Omega) = \frac{2\Omega_H}{M}$ which is smaller than $2\Omega_H$, the Nyquist rate
- Substitute the above expression for $\Omega_T$ in $G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$

Sampling of Bandpass Signals

- This leads to $G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a(j\Omega - j2k(\Delta \Omega))$
- As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_a(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta \Omega$ and scaled by $1/T$
- The amount of shift for each value of $k$ ensures that there will be no overlap between all shifted replicas no aliasing

Sampling of Bandpass Signals

- Figure below illustrate the idea behind
Sampling of Bandpass Signals

• As can be seen, \( g_p(t) \) can be recovered from \( g_s(t) \) by passing it through an ideal bandpass filter with a passband given by \( \Omega_L \leq \Omega \leq \Omega_H \) and a gain of \( T \).

• Note: Any of the replicas in the lower frequency bands can be retained by passing \( g_p(t) \) through bandpass filters with passbands \( \Omega_L - k(\Delta \Omega) \leq \Omega \leq \Omega_H - k(\Delta \Omega) \), \( 1 \leq k \leq M - 1 \) providing a translation to lower frequency ranges.