AN EFFICIENT QMF-WAVELET STRUCTURE

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  - Binomial QMF-Wavelet vs. Discrete Cosine Transform
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Related Work:

R.A. Haddad, "A Class of Orthogonal Nonrecursive Binomial Filters", IEEE Trans. Audio and Electroacoustics, pp. 296-304, Dec. 1971.

A.N. Akansu, "Modified Hermite Transform, A New Orthogonal Transform for SATC of Speech Signals", Ph.D Thesis, Polytechnic Univ., Brooklyn, 1987.

R.A. Haddad and A.N. Akansu, "A New Orthogonal Transform for Signal Coding", IEEE Trans. ASSP, pp.1590-1593, Sept. 1988.

R.A. Haddad and B.G. Nichol, "Efficient Filtering of Images Using Binomial Squences," Proc. ICASSP, pp. 1590-1593, 1989.

R.A. Haddad and A.N. Akansu, "A Class of Fast Gaussian Binomial Filters for Speech and Image Processing", To appear in IEEE Trans. ASSP.

J.B.O.S. Martens, "The Hermite Transform - Theory", To appear in IEEE Trans. ASSP.

J.B.O.S. Martens, "The Hermite Transform - Applications", To appear in IEEE Trans. ASSP.

A.N. Akansu, R.A. Haddad, and H. Caglar, "Perfect Reconstruction Binomial QMF-Wavelet Transform", in preparation.

Hermite Polynomials

The Hermite family of functions are the analog precursors to the digital signals to be presented here. They are briefly reviewed to show similarities between them and their discrete-time counterparts, the Binomial sequences.

The Hermite family of functions is obtained by successive differentiation of the Gaussian $e^{\frac{-t^2}{2}}$

$$u_n(t) = \frac{d^n}{dt^n} e^{\frac{-t^2}{2}} = H_n(t) e^{\frac{-t^2}{2}}$$

$H_n(t)$ are the Hermite polynomials; these can be generated by a two-term recursive formula

$$H_{n+1}(t) + tH_n(t) + nH_{n-1}(t) = 0$$

with

$$H_0(t) = 1 \quad H_1(t) = -t$$

These polynomials also satisfy a second-order differential equation

$$\ddot{H}_n(t) - t\dot{H}_n(t) + nH_n(t) = 0$$

The Hermite family is orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function $e^{\frac{t^2}{2}}$

$$\begin{aligned}\int_{-\infty}^{\infty} u_m(t)u_n(t)e^{\frac{t^2}{2}}dt &= \int_{-\infty}^{\infty} H_m(t)H_n(t)e^{\frac{-t^2}{2}}dt \\ &= (\sqrt{2\pi}n!)\delta_{n-m}\end{aligned}$$

Time-Frequency Relation of Hermite Family:

If $f(t) \leftrightarrow F(w)$ then

$$\frac{d^n}{dt^n} f(t) \leftrightarrow (jw)^n F(w)$$

But $e^{\frac{-t^2}{2}} \leftrightarrow \sqrt{2\pi} e^{-\frac{w^2}{2}}$ and

$$H_r(t) e^{\frac{-t^2}{2}} \leftrightarrow \sqrt{2\pi} (jw)^r e^{-\frac{w^2}{2}}$$

Binomial Family

The discrete counterparts to the Hermite family are generated by successive differences of the Binomial sequence, defined on the interval $[0, N]$

$$X_0(k) = \begin{cases} \begin{pmatrix} N \\ k \end{pmatrix} = \frac{N!}{(N-k)!k!}, & 0 \leq k \leq N \\ 0, & otherwise \end{cases}$$

for N large

$$\begin{pmatrix} N \\ k \end{pmatrix} \sim \frac{2^N}{\sqrt{\frac{N\pi}{2}}} e^{-\{\frac{(k-\frac{N}{2})^2}{\frac{N}{2}}\}}$$

The other members of the Binomial family are obtained from

$$\begin{aligned} X_r(k) &\equiv \triangle^r \binom{N-r}{k-r} \\ &= \nabla^r \binom{N-r}{k}, \quad r = 0, 1, \ldots, N \end{aligned}$$

where

$$\begin{aligned} \triangle f(n) &= f(n+1) - f(n) \quad \text{(forward difference)} \\ \nabla f(n) &= f(n) - f(n-1) \quad \text{(backward difference)} \end{aligned}$$

and

$$\triangle^r f(n-r) = \nabla^r f(n)$$

By taking the successive differences, one obtains

$$\begin{aligned} X_r(k) &= \binom{N}{k} \sum_{\nu=0}^{r} (-2)^\nu \binom{r}{\nu} \frac{k^{(\nu)}}{N^{(\nu)}} \\ &= H_r(k) \binom{N}{k} \end{aligned}$$

where $k^{(\nu)}$ is the forward factorial function, a polynomial in k of degree ν.

$$k^{(\nu)} = \begin{cases} k(k-1)(k-2)\ldots(k-\nu+1), & \nu \geq 1 \\ 1, & \nu = 0 \end{cases}$$

$H_r(k)$ are the discrete Hermite polynomials.

$$X_r(k) = H_r(k) \binom{N}{k}$$

$$\sum_{k=0}^{N} \frac{X_r(k) X_s(k)}{\binom{N}{k}} = \sum_{k=0}^{N} H_r(k) H_s(k) \binom{N}{k}$$

$$= \frac{2^N}{\binom{N}{s}} \delta_{r-s}$$

Binomial Network

$$Z\{X_0(k)\} = \sum_{k=0}^{N} \binom{N}{k} z^{-k} = (1 + z^{-1})^N$$

Also

$$Z\{\nabla^r f(k)\} = (1 - z^{-1})^r Z\{f(k)\}$$

From these properties, it is easy to verify that

$$\begin{aligned} X_0(z) &= \sum_{k=0}^{N} \binom{N}{k} z^{-k} = (1 + z^{-1})^N \\ X_r(z) &= Z\left\{\nabla^r \binom{N-r}{k}\right\} = (1 - z^{-1})^r (1 + z^{-1})^{N-r} \end{aligned}$$

It can also be expressed in the forms

$$\begin{aligned} X_r(z) &= \left(\frac{1-z^{-1}}{1+z^{-1}}\right) X_{r-1}(z) \\ &= \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^r X_0(z) \qquad \begin{pmatrix} \text{Binomial} \\ \text{Network} \end{pmatrix} \end{aligned}$$

In the time domain,

$$X_{r+1}(k) = -X_{r+1}(k-1) + X_r(k) - X_r(k-1) \qquad 0 \le k, \;\; r \le N$$

with $X_r(-1) = 0 \quad 0 \le r \le N \quad X_0(k) = \begin{pmatrix} N \\ k \end{pmatrix}$

<u>Frequency Response of Binomial Filters</u>

$$\begin{aligned} z = e^{jwT} &= e^{j\theta} \\ \theta &= wT(\text{normalized frequency}) \\ X_0(z)\,|_{z=e^{j\theta}} \equiv X_0(\theta) &= (1+e^{-j\theta})^N \\ &= (2e^{-j\frac{\theta}{2}}\cos\frac{\theta}{2})^N \\ &= 2^N e^{-jN\frac{\theta}{2}}(\cos\frac{\theta}{2})^N \end{aligned}$$

the amplitude and phase responses respectively

$$\begin{aligned} A_0(\theta) &= 2^N(\cos\frac{\theta}{2})^N \\ \psi_0(\theta) &= -\frac{N\theta}{2} \end{aligned}$$

(The Binomial filter is a good approximation to a low-pass narrow-band Gaussian filter)

Similarly, for $r \geq 1$

$$X_r(z) = (1 - z^{-1})^r (1 + z^{-1})^{N-r}$$

and $z = e^{j\theta}$

$$X_r(\theta) = A_r(\theta) e^{j\psi_r(\theta)}$$

where

$$\begin{aligned} A_r(\theta) &= 2^N (\sin\frac{\theta}{2})^r (\cos\frac{\theta}{2})^{N-r} \\ \psi_r(\theta) &= \frac{r\pi}{2} - \frac{N\theta}{2} \end{aligned}$$

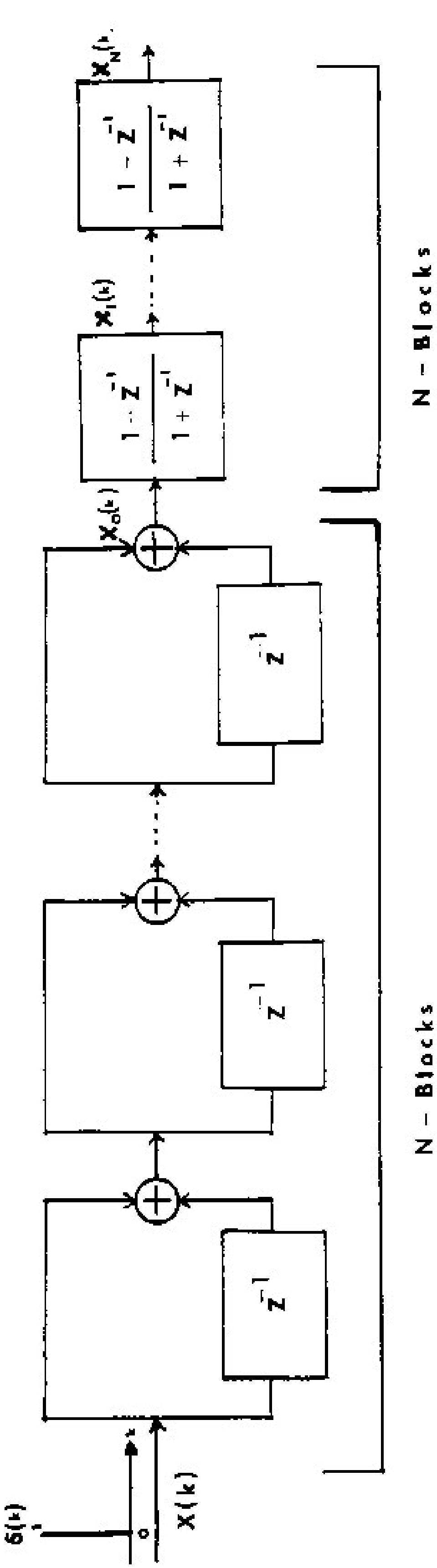
The center frequency of the band r is found to be

$$\theta_m = 2\cos^{-1}(\sqrt{1 - \frac{r}{N}}) = 2\sin^{-1}(\sqrt{\frac{r}{N}})$$

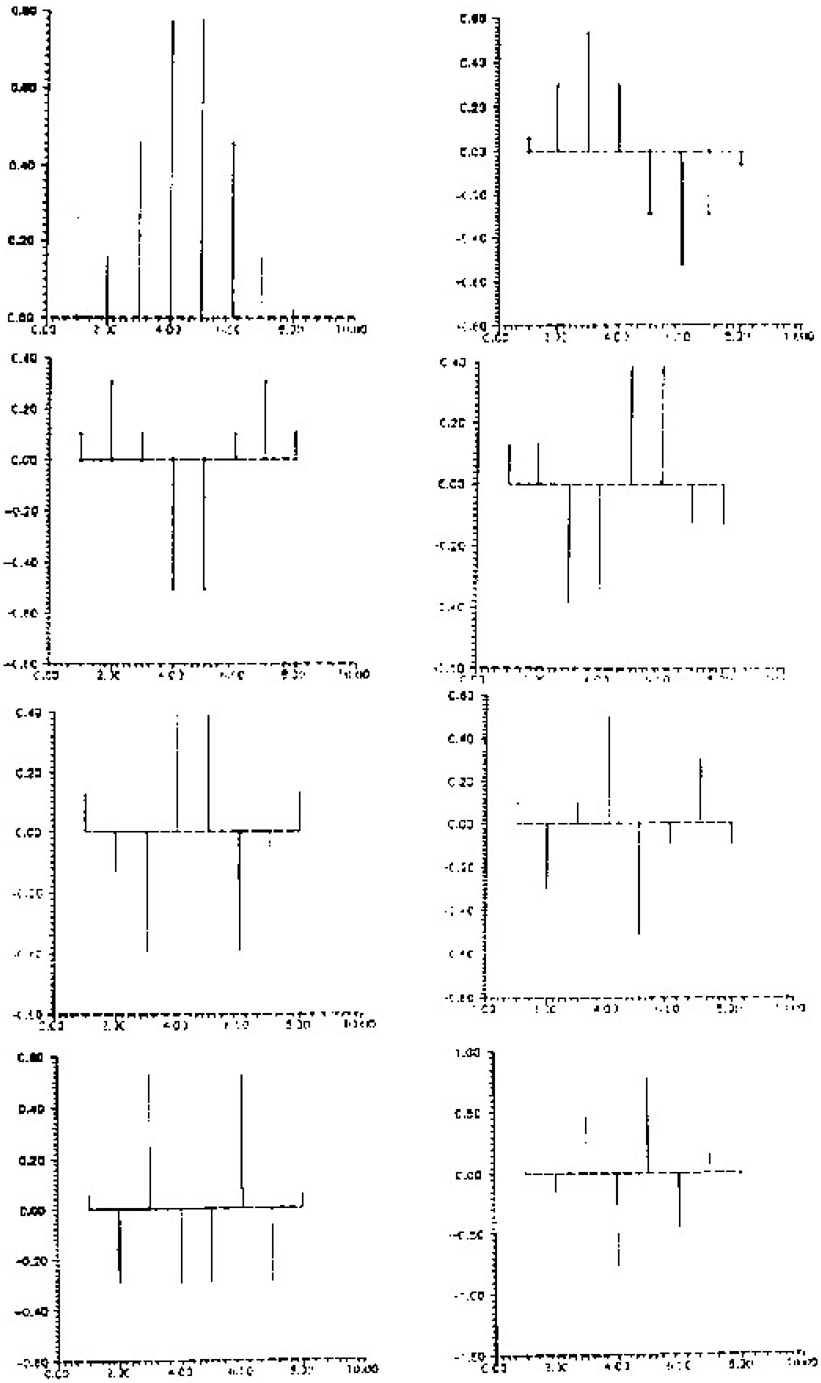
and the corresponding maximum band value

$$A_r(\theta_m) = (2)^N (\frac{r}{N})^{\frac{r}{2}} (1 - \frac{r}{N})^{\frac{(N-r)}{2}}$$

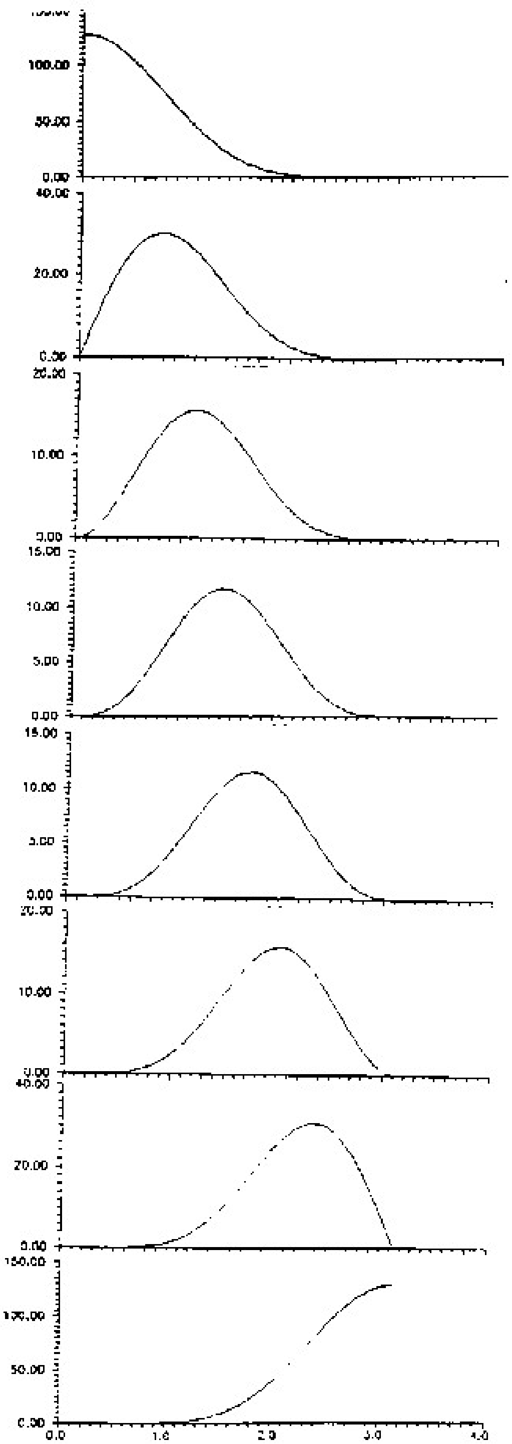
* Note that there is no multiplication operation in the Binomial Network (Filter Bank).



BINOMIAL NETWORK



BINOMIAL SEQUENCES
Length 8



MAGNITUDE RESPONSES
OF BINOMIAL SEQUENCES
LENGTH 8

Binomial QMF Structure

Question: Can we obtain a half-band filter by linearly combining low-frequency sequences of the Binomial family which satisfies QMF magnitude condition?

Find an $h_0(n)$ function as

$$h_0(n) = \sum_{r=0}^{\frac{N-1}{2}} \theta_r X_r(n) \qquad n = 0, 1, \ldots, N \qquad ; \qquad N \text{ odd}$$

which satisfies the magnitude condition in frequency

$$\mid H_0(e^{jw}) \mid^2 + \mid H_0(e^{j(w+\pi)}) \mid^2 = C$$

where C is a constant. In time domain, this becomes

$$\sum_{i=0}^{N} h_0(n)h_0(n+2i) = \delta_{0i} \quad i = 0, \ldots, N$$

Perfect Reconstruction Requires

$$\begin{aligned}
T(z) &= H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = C \\
T_1(z) &= H_0(z)H_0(z^{-1}) \\
T_2(z) &= T_1(-z) \\
& M = N \qquad = \text{odd}
\end{aligned}$$

$T_1(z)$ is the power spectral density function of $H_0(z)$. For $h_0(n)$ causal, length M

$$\begin{aligned}
T_1(z) &= \gamma_M z^M + \gamma_{M-1} z^{M-1} + \ldots + \gamma_0 + \gamma_1 z^{-1} + \ldots + \gamma_M z^{-M} \\
T_2(z) &= -\gamma_M z^M + \gamma_{M-1} z^{M-1} + \ldots + \gamma_0 - \gamma_1 z^{-1} + \ldots - \gamma_M z^{-M} \\
T_1(z) + T_2(z) &= \ldots + \gamma_2 z^2 + \gamma_0 + \gamma_2 z^{-2} + \ldots = \frac{C}{2}
\end{aligned}$$

To make $T(z) =$ constant, it is sufficient to make

$$\gamma_{2n} = 0; \qquad n = 1, 2, \ldots, \frac{M-1}{2}$$

$$\begin{aligned} H_0(z)H_0(z^{-1}) &= T_1(z) = R(z) \leftrightarrow \rho(n) = \sum_{k=0}^{M} h_0(k)h_0(k+n) \\ R(z) &= \gamma_M z^M + \gamma_{M-1} z^{M-1} + \ldots + \gamma_0 + \ldots + \gamma_M z^{-M} \end{aligned}$$

need to force even-indexed coefficients in $R(z) = 0$, $n \neq 0$

$$\rho(2n) = 0 \ ; \quad n = 1, 2, \ldots, \frac{M-1}{2}$$

normalized with $\rho(0) = 1$

$$
\begin{aligned}
h_0(k) &= \theta_0 X_0(k) + \ldots + \theta_{\frac{M-1}{2}} X_{\frac{M-1}{2}}(k) \\
\rho(n) &= \sum_{k=0}^{M} h_0(k) H_0(k+n) \\
&= \sum_{r=0}^{\frac{M-1}{2}} \sum_{s=0}^{\frac{M-1}{2}} \theta_r \theta_s \left(\sum_{k=0}^{M} X_r(k) X_s(k+n) \right) \\
\rho_{rs}(n) &= \sum_{k=0}^{M} X_r(k) X_s(k+n) \\
\rho(n) &= \sum_{r=0}^{\frac{M-1}{2}} \sum_{s=0}^{\frac{M-1}{2}} \theta_r \theta_s \rho_{rs}(n)
\end{aligned}
$$

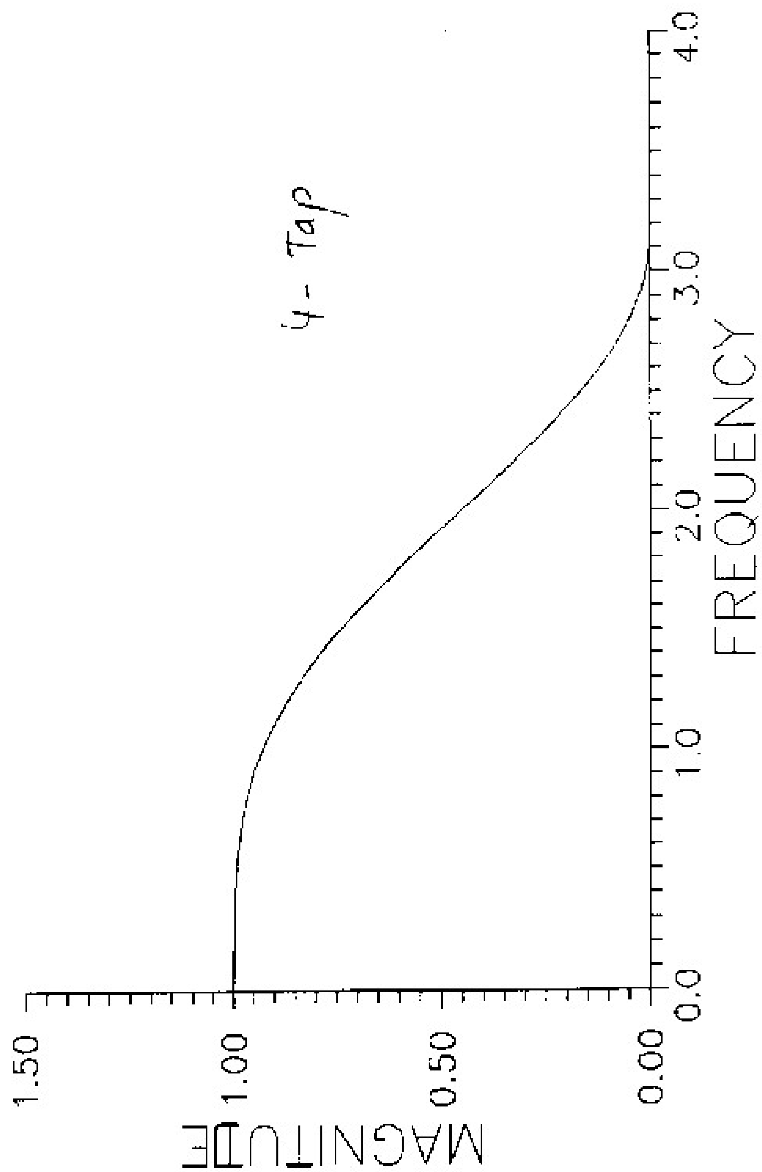
Find $\{\theta_r\}$ satisfies $\rho(2n) = 0; \;\; n = 1, 2, \ldots, \frac{M-1}{2}$ with $\rho_0 = 1$

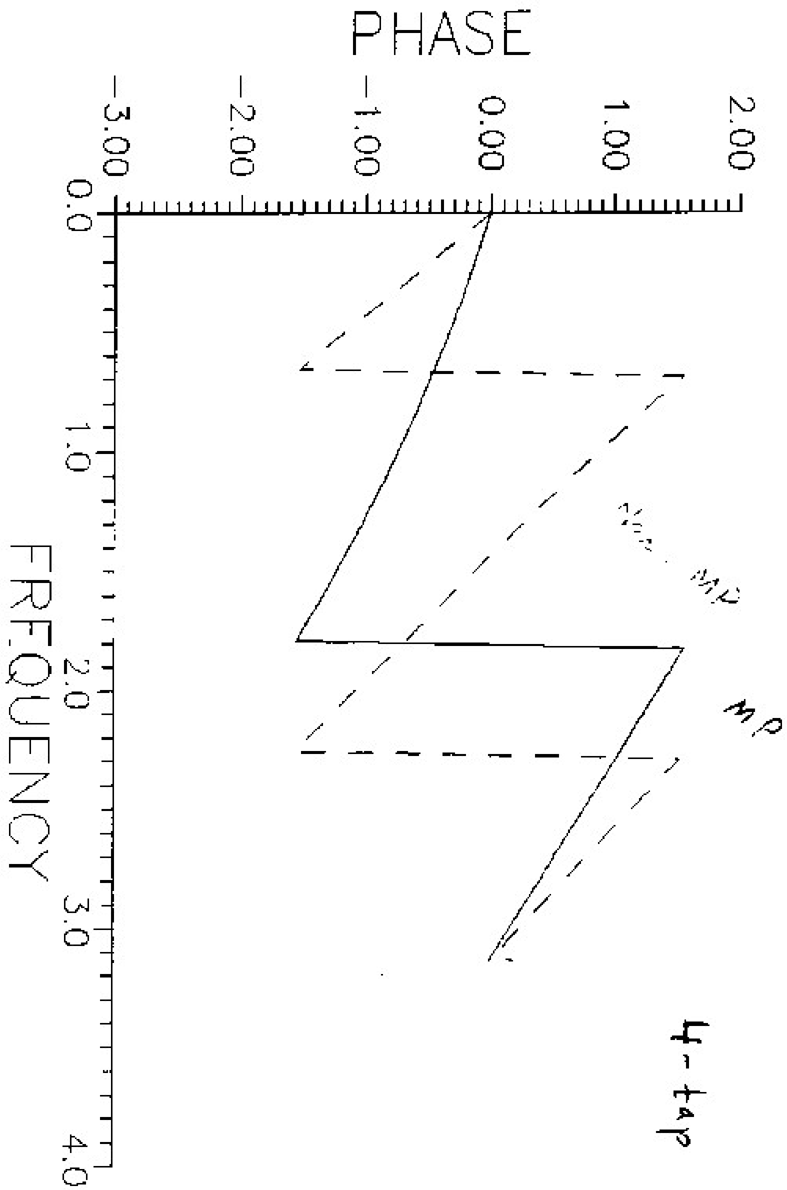
- There are a set of coefficients which satisfy the squared magnitude condition for any N, N is odd.
- One set of coefficients provides a minimum-phase solution.

N=3		
θ_r	set 1	set 2
θ_0	1	1
θ_1	$\sqrt{3}$	$-\sqrt{3}$

N=5		
θ_r	set 1	set 2
θ_0	1	1
θ_1	$\sqrt{2\sqrt{10}+5}$	$-\sqrt{2\sqrt{10}+5}$
θ_2	$\sqrt{10}$	$\sqrt{10}$

N = 7				
θ_r	set 1	set 2	set 3	set 4
θ_0	1	1	1	1
θ_1	4.9892	-4.9892	1.0290	-1.0290
θ_2	8.9461	8.9461	-2.9705	-2.9705
θ_3	5.9160	-5.9160	-5.9160	5.9160


4- Tap
MAGNITUDE
1.50
1.00
0.50
0.00
FREQUENCY
0.0
1.0
2.0
3.0
4.0



PHASE
-3.00
-2.00
-1.00
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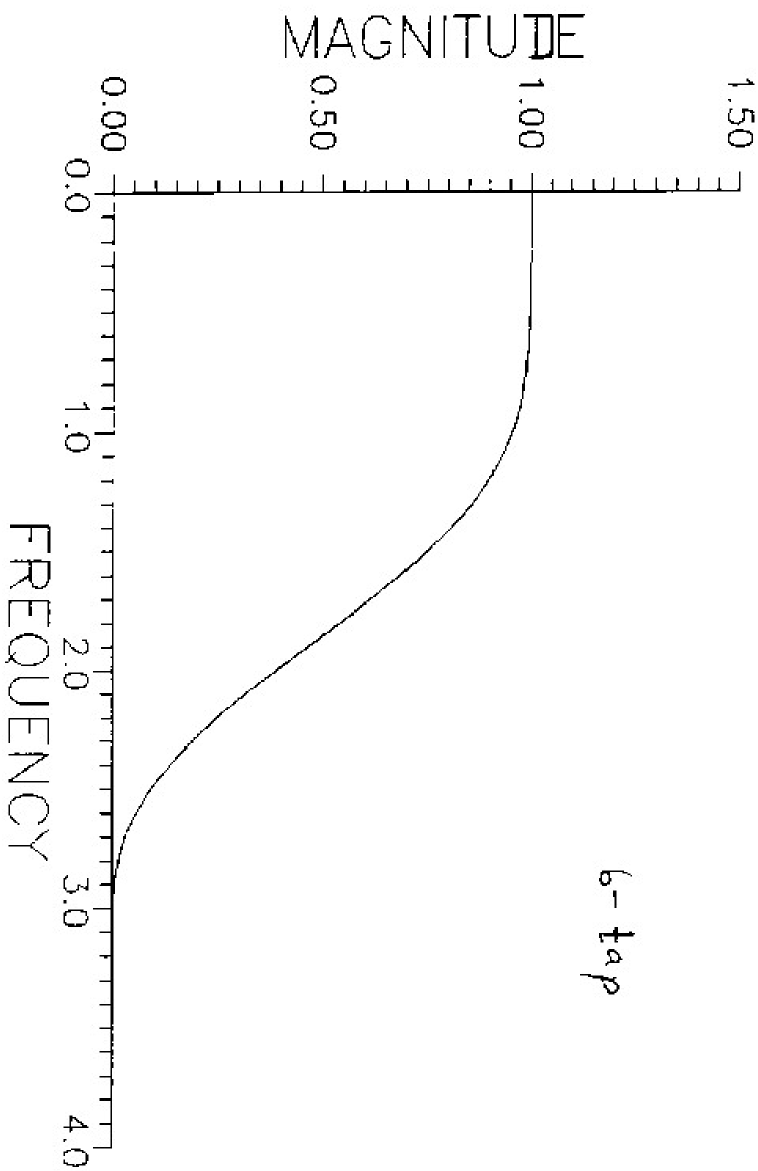
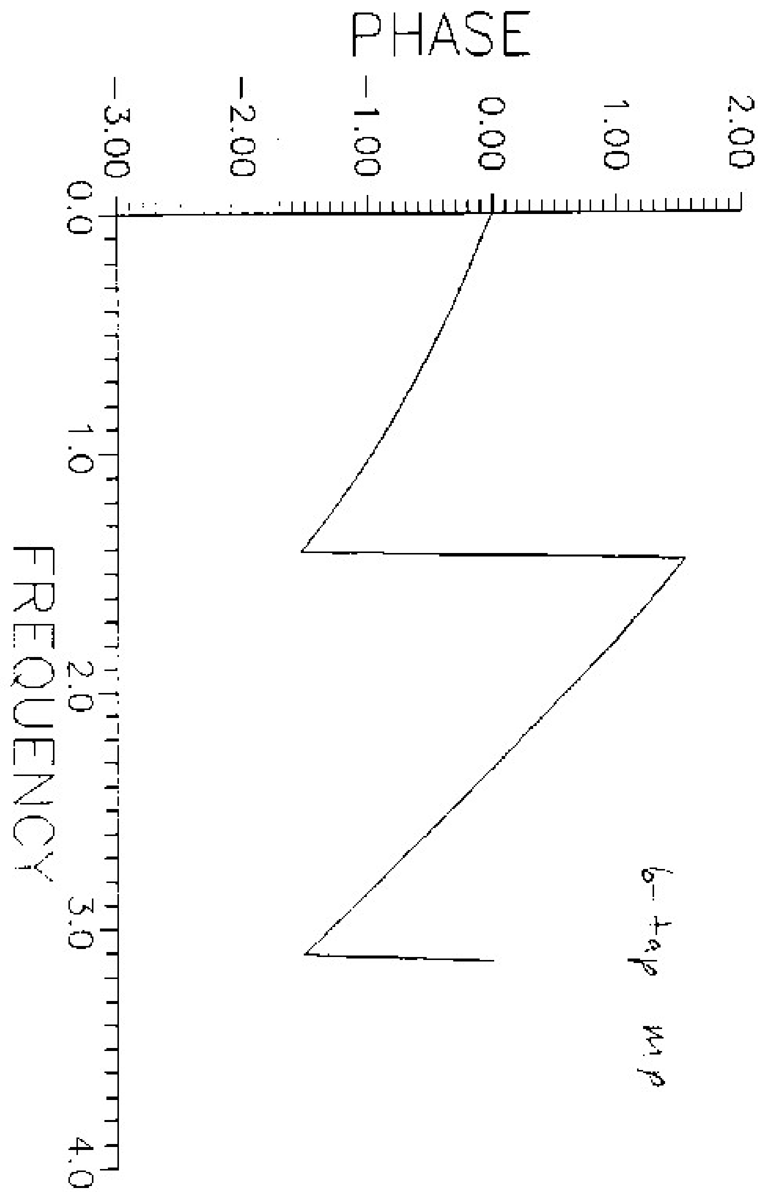
1.00
2.00
0.0
1.0
2.0
3.0
4.0
FREQUENCY
MP
4-tap



TAP 6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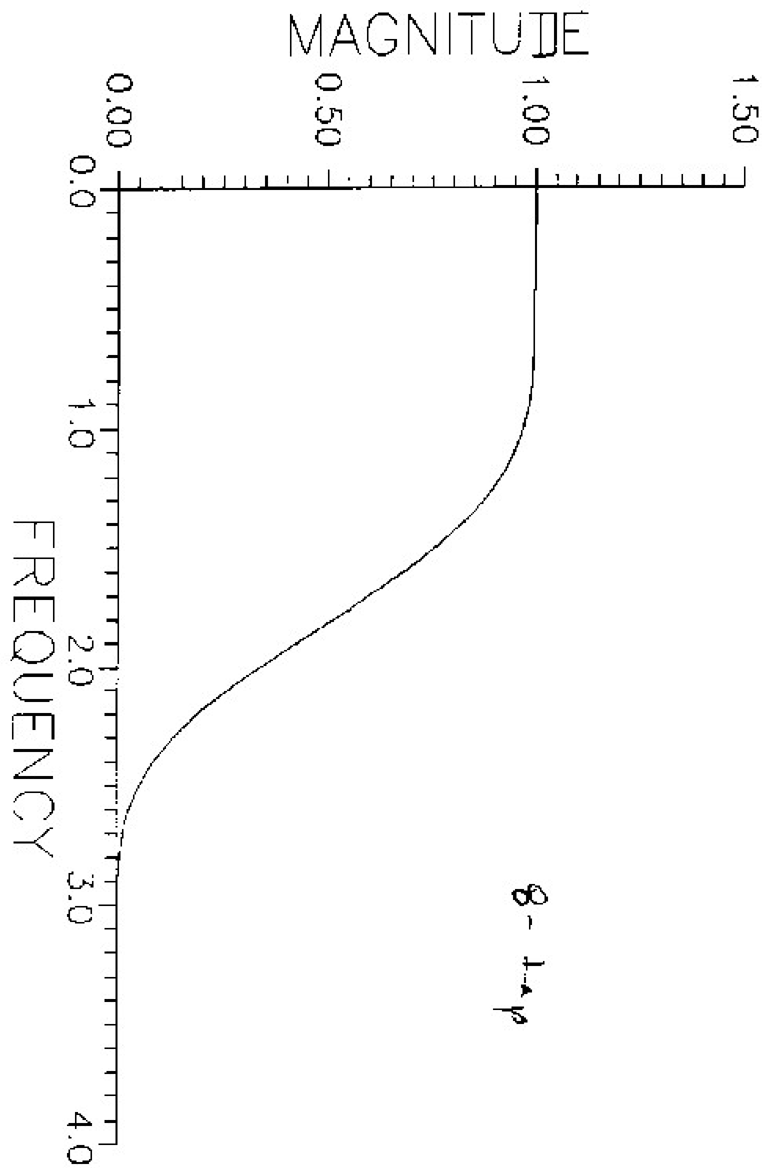
FREQUENCY


MAGNITUDE
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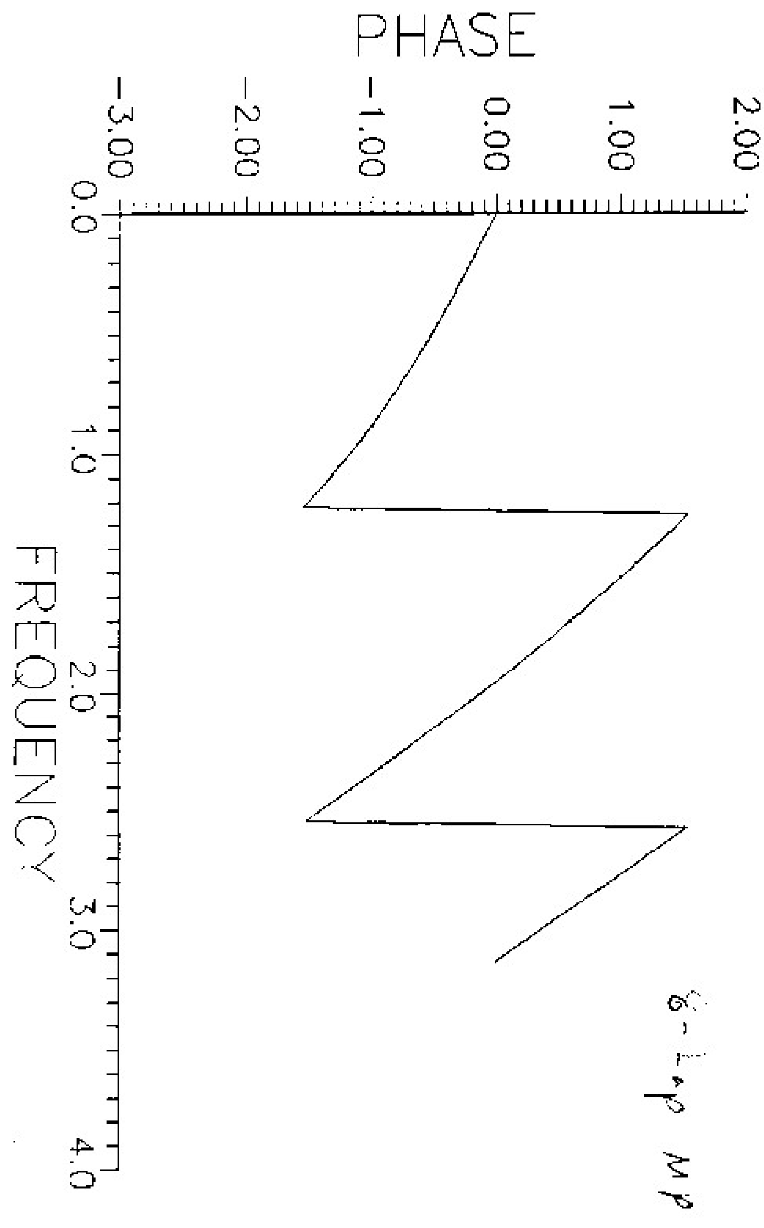
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FREQUENCY
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4.0

Remarks on Phase Response

- Note that no phase constraint is imposed in the problem!
- $h(n)$ does not have a linear phase response for real θ_r. Fortunately, $h(n)$ has almost-linear phase response. (Practically linear)
- Non-minimum phase coefficient sets provide even more almost-linear phase responses.
- Phase response of $h_0(n)$ for $N = 3, 7$; (minimum phase solutions)

Connections to Wavelet Transform

- Additional to Perfect Reconstruction Conditions, (Conventional QMF), the Wavelct approach provides regularity condition.

- This condition provides maximally flat pass band and stopbands.

- The regularity condition requires the low-pass filter has a sufficient number of zeros at $z = -1$.

- Binomial QMF satisfies the regularity and wavelet conditions.

- Binomial QMF filters are identical with the filters of orthonormal wavelets derived in

  I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets", Commun. on Pure and Applied Mathematics, Vol. XLI 909-996, 1988.

Performance Comparisons

- Binomial QMF-Wavelet vs Discrete Cosine Transform

  a. Source Models (1-D)

  b. Real Images (Lena, Building, Cameraman, Brain)

a Auto-regressive, order 1, AR(1) source models are crude approximations to the natural speech and image sources. 1-D AR(1) source is defined as

$$X(n) = \rho X(n-1) + \xi(n)$$

where ρ is the correlation coefficient and $\xi(n)$ is white noise with known power.

The performance measure used here is the gain of transform coding over PCM, G_{TC} is defined as

$$G_{TC} = \frac{\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2}{\left[\prod_{i=1}^{N}\sigma_i^2\right]^{\frac{1}{N}}}$$

where σ_i^2 is the variance of the i^{th} coefficient, and

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 \quad \text{(variance preserving decomposition)}$$

Similarly, gain of subband coding over PCM;

$$G_{SBC} = \frac{\frac{1}{N}\sum_{l=1}^{N}\sigma_l^2}{\left[\prod_{l=1}^{N}\sigma_l^2\right]^{\frac{1}{N}}}$$

where σ_l^2 is the variance of the l^{th} band and

$$\sigma_x^2 = \frac{1}{N}\sum_{l=1}^{N}\sigma_l^2$$

*G_{TC} and G_{SBC} assume that all the bands have the same type of pdfs.

	ρ	G_{TC}	G_{sbc} 4-tap	6-tap	8-tap	16-tap
4 × 4 Trans.	0.95	5.71	6.43	6.77	6.91	7.08
or	0.85	2.59	2.82	2.95	3.01	3.07
	0.75	1.84	1.95	2.02	2.05	2.09
4-band QMF	0.65	1.49	1.56	1.60	1.62	1.64
(2 levels)	0.5	1.23	1.26	1.28	1.29	1.30
8 × 8 Trans.	0.95	7.63	10.96	12.76	13.99	16.97
or	0.85	3.03	4.18	4.82	5.27	6.36
	0.75	2.03	2.73	3.10	3.37	4.05
8-band QMF	0.65	1.59	2.10	2.36	2.55	3.04
(3 level)	0.5	1.27	1.66	1.84	1.97	2.32

b. Performance results for real 2-D sources

- Separable transform and QMF structure.

		G_{TC}	G_{SBC} 4-tap	G_{SBC} 6-tap	G_{SBC} 8-tap
4×4 2-D Trans.	LENA	16.002	16.70	18.99	20.37
or	BUILDING	14.107	15.37	16.94	18.17
16-Band Regular	CAMERAMAN	14.232	15.45	16.91	17.98
Tree	BRAIN	3.295	3.25	3.32	3.42
8×8 2-D Trans.	LENA	21.988	19.38	22.12	24.03
or	BUILDING	20.083	18.82	21.09	22.71
64-Band Regular	CAMERAMAN	19.099	18.43	20.34	21.45
Tree	BRAIN	3.788	3.73	3.82	3.93

- An efficient QMF-Wavelet structure is proposed. (Same filters derived by Daubechies)

- Most serious competitor to DCT for coding applications
  - Very simple to implement on VLSI
  - Outperforms DCT

- Almost-linear (practically linear) phase response

- Linear vs Almost-linear phase response for future work! (symmetric vs. almost symmetric or symetric-like unit sample response)

- We also derived multi-band perfect reconstruction filter banks and biorthogonal wavelets similarly