Wavelets: A different way to look at subband coding.

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"Wavelets"

Technique to cut up \{ data functions \}
operators

into different frequency components, and
to study each component with a resolution
matched to their scale.

This technique was "invented" independently
in several different fields

- pure mathematics: harmonic analysis
  (Calderón)
- quantum mechanics: coherent states
  (Ashtakoon- Kleindler)
- engineering: signal analysis
  (SHF filters - Botebain & Galland
  Smith & Barnwell
  Jean Morlet )

Recently (last three years): synthesis between
different approaches → very fertile for all branches.
ORTHONORMAL BASES OF WAVELETS.

Old example: Haar basis.

\[
\begin{align*}
&\begin{array}{c}
\text{support } \Phi_0 = [-1, 1] \\
\text{support } \Phi_{j,k} = \left(2^{-j}, 2^{-j}\right)
\end{array} \\
\text{center of support } \Phi_{j,k} = (k+1/2)2^{-j}
\end{align*}
\]

Proof that \( \Phi_{j,k} \) constitute orthonormal basis?

Sufficient to show that functions with support in \([-2^j0, 2^j0]\), preciseness constant on intervals \([2^j-2^{-j}, (k+1)2^{-j}]\), can be written as combination of \( \Phi_{j,k} \).
\[
\sum_{k=1}^{K} 2^{-k} a_k + (2^{-j_0-k} x) + 2^{-K} a \phi(2^{-j_0-K} x)
\]

where \( \phi \) is
\[
\begin{array}{c}
1 \\
-1
\end{array}
\]

But \( \|2^{-K} a \phi(2^{-j_0-K} x)\|^2 \leq 2^{-2K} (a_k^2) 2^{-j_0+K} \to 0 \) for \( K \to \infty \).

\Rightarrow \text{done!}

In fact, proof uses \underline{multiresolution analysis}:
- introduces "averaging" function \( \phi \)
- space \( V_j \) spanned by \( \phi(2^{-j} x - k) \)
  \( V_j \subset V_{j-1} \)
- \( \text{Proj}_{V_j} f = \text{Proj}_{V_{j-1}} f + \text{expansion in the } \{y_k\} \).
Orthogonal bases of wavelets.

For very special $\psi$:
$$2^{-m/2} \psi(2^{-m} t - n) = \psi_{mn}(t)$$
are orthonormal basis.

NB. $b_0 = 1$ is not really a restriction.
$a_0 = 2$ : computationally easy.
other $a_0$ also possible. (in fact
all rational values are allowed)

These are associated to a beautiful mathematical construction:

Multi-resolution analysis. (S. Mallat,
Y. Meyer)

ladder of spaces

$$\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$$

$$\forall m \in \mathbb{Z}$$

$$\bigcup_{m \in \mathbb{Z}} V_m = L^2(\mathbb{R})$$

$V_0$ describes functions in which all scales finer than $2^0$ are left out.

$$2x: \quad V_0 \quad - \quad V_1 \quad - \quad V_2 \quad - \quad V_3$$
general framework for construction of orthonormal wavelet bases.

\[
\cdots, V_1, V_0, V_{-1}, V_{-2}, \ldots
\]

\[
\bigoplus_{d \in \mathbb{Z}} \mathcal{V}_d = L^2(\mathbb{R}).
\]

\[
\mathcal{V}_d \ni f \iff f(2^d \cdot) \in \mathcal{V}_0.
\]

\[
\exists \phi \in \mathcal{V}_0 \text{ so that } \phi_{0,k} \text{ are o.n. basis for } \mathcal{V}_0
\]

\[
\phi_{0,k}(x) = \phi(x - k).
\]

MULTIRESOLUTION ANALYSIS.

Then, \exists associated orthonormal wavelet basis.

\[
\mathcal{W}_0 : \text{orthogonal complement in } V_1 \text{ of } \mathcal{V}_0
\]

\[
\mathcal{V}_0 \oplus \mathcal{W}_0 = V_1, \quad \mathcal{W}_0 \perp \mathcal{V}_0.
\]

\[
\exists \Psi \in \mathcal{W}_0 \text{ so that } \Psi_{0,k} \text{ are orthonormal basis for } \mathcal{W}_0
\]

\[
\text{Proj}_{V_1} = \text{Proj}_{\mathcal{V}_0} + \text{expansion in } \Psi_{0,k}.
\]

\[
\cdots, \mathcal{V}_1, \mathcal{V}_0, \mathcal{V}_{-1}, \mathcal{V}_{-2}, \ldots
\]

\[
\mathcal{W}_1 \subset \mathcal{W}_0 \subset \mathcal{V}_{-1} \subset \mathcal{V}_{-2} \subset \mathcal{W}_{-2} \subset \mathcal{W}_{-3}
\]

\[
f \in \mathcal{W}_d \implies f(2^d x) \in \mathcal{W}_0.
\]

\[
\forall \mathcal{W}_i \text{ all orthogonal, and } \bigoplus_{i=-1}^{-\infty} \mathcal{W}_i = L^2(\mathbb{R}).
\]
\[ w_j \text{ dilated version on } W_0 \]
\[ \text{on. basis } \mathcal{V}(x-k) \]
\[ \Rightarrow \mathcal{V}_j(x) = 2^{-j/2} \mathcal{V}(2^{-j}x-k), \quad k \in \mathbb{Z} \]
\[ \text{on. basis in } W_j \]
\[ \Rightarrow \{ \mathcal{V}_j : j \in \mathbb{Z} \} \text{ on. basis for } L^2(\mathbb{R}) \]

Recipe for \( \mathcal{V} \):

1. \( \phi \in V_0 \subset V_{-1} \rightarrow \text{on. basis } \phi_{-1,n} \)
2. \[ \phi(x) = \sum_n h_n \phi_{-1,n}(x) \]
   \[ = \sqrt{2} \sum_n h_n \phi(2x-n) \]
   \[ h_n = \langle \phi, \phi_{-1,n} \rangle \]
3. \[ \mathcal{V}(x) = \sqrt{2} \sum_n (-1)^n h_{-n+1} \phi(2x-n) \]

To prove existence + recipe for \( \mathcal{V} \), analyze in detail what \( W_0 \) really represents.

A crucial role in this analysis is played by the trigonometric polynomial
\[ m_0(\xi) = \frac{1}{\sqrt{2}} \sum_n h_n e^{-2\pi i n \xi} \].
\[ |m_0(\xi)|^2 + |m_0(\xi+\frac{1}{2})|^2 = 1. \]
to generalize Haar basis. generalize the associated multiresolution analysis.

Two paths.

- Generalize
  - Precise constant
    - Linear
    - Quadratic
    - Cubic (splines)

But \( \phi \) is not orthonormal!

Lo orthonormalization trick.

\[
\tilde{\phi}(\xi) = \frac{\phi(\xi)}{\left( \sum_k |\phi(k\xi)|^2 \right)^{1/2}}
\]

\( \tilde{\phi} \) is orthonormal.

span same space as \( \phi \)

\( \phi \) can be used to construct \( \tilde{\phi} \).

orthonormalization trick loses compact support wanted arbitrarily high regularity. only finite \# of \( h_n \) allowed.

\[
m_0(\xi) = \frac{1}{12} \sum h_n e^{-2\pi i n \xi}
\]

\[
1 m_0(\xi) + 1 m_0(\xi + 1/2) = 0
\]

\[
\phi(\xi) = m_0(\xi/2) \tilde{\phi}(\xi/2)
\]

\[
\prod_{\text{d} = 1} m_0(2^{-d} \xi)
\]

- Strategy to construct \( m_0 \) so that infinite product has decay
  - check that strategy works!
ONDELETTE:
Wanted: \[ m_0(\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \eta_n e^{-2\pi i n \xi} \]

\[ |m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 = 1 \]

\[ \prod_{j=1}^{\infty} m_0(2\pi j \xi) \text{ decays for } |\xi| \to \infty. \]

\[ m_0(\xi) = \left( \frac{1 + e^{-2\pi i \xi}}{2} \right)^L \xi(\xi) \]

\[ \frac{1 + e^{-2\pi i \xi}}{2} = e^{-\pi i \xi} \cos \pi \xi \]

\[ \prod_{j=1}^{\infty} \cos \left( 2\pi j \xi \right) = \frac{\sin \pi \xi}{\pi \xi} \]

\[ \Rightarrow \text{ if sufficient control over } \prod_{j=1}^{\infty} \xi(2\pi j \xi), \]

\[ \text{then } \xi \text{ will have good decay.} \]

\[ (\cos \pi \xi)^{2L} |\xi(\xi)|^2 + (\sin \pi \xi)^{2L} |\xi(\xi + 1/2)|^2 = 1 \]

polynomial in \( \cos 2\pi \xi \)

\[ \Rightarrow \text{polynomial in } \sin^2 \pi \xi \]

\[ (1 - y)^L P(y) + y^L P(1-y) = 1 \]

\[ P(y) = \frac{1}{(1-y)^L} + O(y^L) \]

\[ = \sum_{l=0}^{L-1} \left( \frac{L-1}{l} \right) y^l + O(y^L) \]
\[
\Rightarrow (\mathfrak{F}(\xi))^2 = \sum_{l=0}^{L-1} (L-l) (\sin \pi \xi l)^2
\]

- Use a lemma by Rice to "extract square root"

\[
\Rightarrow \mathfrak{F}(\xi) = \sum_{l=0}^{L-1} \mathcal{F}_l e^{-2\pi i \xi l}
\]

where real.

- \[\left| \mathfrak{F}(2^{-j} \xi) \right| \leq C (1 + |\xi|)^{\mu - j} \quad \mu \geq 0.81 \]

- \[\left| \mathfrak{F}(\xi) \right| \leq C (1 + |\xi|)^{-\nu - j} \quad \nu \geq 0.19 \]

* arbitrarily high regularity!
What does all this have to do with subband coding?

\[ f \in V_0 \quad f = \sum_{n} b_n \phi_{on} \]

\[ V_1 \oplus W_1 \]

\[ f = s + d \quad s = \sum_{k} s_k \phi_{1k} \]

\[ d = \sum_{k} d_k \psi_{1k} \]

\[ \phi_{1k} = \sum_{n} h_{n-2k} \phi_{on} \]

\[ \psi_{1k} = \sum_{n} g_{n-2k} \phi_{on} \]

\[ g_n = (-1)^n h_{-n+1} \]

\[ \Rightarrow \quad s_k = \frac{1}{n} h_{n-2k} f_n \]

\[ d_k = \frac{1}{n} g_{n-2k} f_n \]

Inverse transform: uses transposed matrix

\[ f_m = \sum_{k} \left[ h_{m-2k} s_k + g_{m-2k} d_k \right] \]

Diagram of subband coding with exact reconstruction.
the "filter coefficients" \( h_n \), \( g_n \) coming from an orthonormal basis of wavelets correspond exactly to the filters in an exact reconstruction subband coding scheme.

( \( \rightarrow \) compact support for \( \phi, \psi \) important! leads to FIR filters).

What role does regularity play?

Suppose you have the decomposition

\[
\begin{align*}
\mathcal{F} & \rightarrow \mathcal{F}_L \rightarrow \mathcal{F}_{LL} \rightarrow \mathcal{F}_{LLL} \rightarrow \\
& \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& \mathcal{F}_H \quad \mathcal{F}_{LH} \quad \mathcal{F}_{LHH} \\
\end{align*}
\]

with reconstruction

\[
\begin{align*}
\mathcal{F}_L & \oplus \mathcal{F}_L \oplus \mathcal{F}_{LL} \oplus \mathcal{F}_{LLL} \quad \leftarrow \\
& \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
& \mathcal{F}_H \quad \mathcal{F}_{LH} \quad \mathcal{F}_{LHH} \\
\end{align*}
\]

What does a sequence

\[
\mathcal{F}_{LLL} = 0 0 0 1 0 0 0 
\]

correspond to?
\[
\begin{align*}
(f_0)_n &= \delta_{h_0} \\
(f_j)_h &= \frac{1}{n} \sum_{h\leq k} (f_{j-1})_n \\
F_d(x) &\text{ piecewise constant on } [\frac{d}{2-\delta}, (d+1)\delta] \\
F_d^n(x) &= \frac{1}{\sqrt{2}} \sum_{n} h_n F_{d-1} \left(2x-n\right) \\
F_d &\rightarrow \text{ fixed point of } T \\
(\text{IF})_d(x) &= \frac{1}{\sqrt{2}} \sum_{n} h_n F(2x-n) \\
\text{But this fixed point is } \phi ! \\
\phi(x) &= \frac{1}{\sqrt{2}} \sum_{n} h_n \phi(2x-n)
\end{align*}
\]
Regularity is a good idea.

Regularity forces constraints on filters.

\[ \phi, \psi \in C^k \]

\[ m_0(f) = \left( \frac{1 + e^{-2\pi i f}}{2} \right) \mathbb{I}(f) \]

Low-pass filter has zero of order \( k \)
at \( f = \frac{1}{2} \).

Same is true for generalizations of orthonormal wavelet bases to higher dimensions, or to other dilation factors than 2.
Biorthogonal wavelet bases.

\[ \phi = \sum_{d,k} <\phi, \Psi_{d,k} > \Psi_{d,k} \]

- correspond to analysis filters
- synthesis filters

- symmetric \( \Psi \) possible (\( \rightarrow \) linear phase filters?)

- regularity constraints:
  \[ \Psi \in \mathcal{C}^k \Rightarrow \int dx x^l \tilde{\Psi}(x) = 0 \quad l=0, \ldots, k-1 \]

  \[ \Rightarrow m_0(x) = \left( \frac{1 + e^{-2\pi i x}}{2} \right)^k \tilde{\Psi}(x) \]

  regularity on both \( \Psi, \tilde{\Psi} \)

  \( \Rightarrow \) both \( m_0, \tilde{m}_0 \) need factorization of this kind.

- Examples:
  - \( m_0 \) binomial (\( \Rightarrow \) B-spline)
    - family of more and more regular \( m_0 \)
  - rearrange previous examples (cf. Vetterli)
  - \( m_0, \tilde{m}_0 \) both very close to orthonormal case

\[ m_0(x) = -0.5e^{-2\pi i x} + 0.25e^{-4\pi i x} + 0.25e^{-6\pi i x} - 0.5e^{-8\pi i x} \]
"Wavelets" : more than just orthonormal bases!

**Continuous case**

\[ y_{a,b}(x) = \frac{1}{\sqrt{a}} y\left( \frac{x-b}{a} \right) \]

\[ \int_{-\infty}^{\infty} y(x) \, dx = 0 \quad y \text{ symmetric} \]

\[ f = C_4^{-1} \int_{0}^{a} \int_{b}^{\infty} \, db \quad <f, y_{a,b}> = y_{a,b} \]

\[ <f, y_{a,b}> = \int_{-\infty}^{\infty} f(t) \cdot y_{a,b}(t) \, dt \]

**Frames**

\[ a = a_0^m \quad m \in \mathbb{Z} \quad (a_0 > 1 \text{ fixed}) \]

\[ b = n b_0 a_0^m \quad n \in \mathbb{Z} \quad (b_0 > 0 \text{ fixed}) \]

\[ y_{mn}(x) = a_0^{-m/2} y\left( a_0^{-m} x - n b_0 \right) \]

2 dual points of view:

- characterize \( f \) by \( <f, y_{mn}> \)
- find \( a_{mn} \) so that \( f = \sum_{m,n} a_{mn} y_{mn} \)
CONCLUSION.

- Subband coding with exact reconstruction

  $\leftrightarrow$ anormal wavelet bases
  biorthogonal wavelet bases

  Regularity!

- More to wavelets than subband coding.