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Generalized Discrete Fourier Transform: Theory and Design Methods

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ABSTRACT

Constant amplitude transforms like discrete Fourier transform (DFT), Walsh transform, nonlinear phase Walsh-like transforms and Gold codes have been successfully used in many wire-line and wireless communications technologies including code division multiple access (CDMA), discrete multi-tone (DMT), and orthogonal frequency division multiplexing (OFDM) types. In this paper, we present a generalized framework for DFT called Generalized DFT (GDFT) with nonlinear phase by exploiting the phase space. It is shown that GDFT offers sizable performance improvements over Walsh, Gold and DFT codes in multi-carrier communications scenarios considered. We also highlight that known constant modulus code families are special solutions of the proposed GDFT framework. Moreover, we introduce practical design methods offering computationally efficient implementations for GDFT. We expect performance improvements in future communications systems employing GDFT intelligently.

***Index Terms*— Discrete Fourier Transform, Generalized Discrete Fourier Transform, OFDM, DMT, Walsh Codes, Gold Codes.**

I. MATHEMATICAL PRELIMINARIES

An N^{th} root of unity is a complex number satisfying the equation

$$z^N = 1 \quad N = 0,1,2,... \tag{1}$$

If z holds Eq. (1) but $z^m \neq 1$; $0 < m < N-1$, then z is defined as a *primitive* N^{th} *root of unity*. The complex number $z_0 = e^{j(2\pi/N)}$ is the primitive N^{th} root of unity with the smallest positive argument. The other primitive N^{th} roots of unity are expressed as

$$z_k = e^{j(2\pi/N)k} \quad k = 1,2,3, \ ... \ , N-1 \tag{2}$$

where k and N are *co-prime*. All *primitive* N^{th} *roots of unity* satisfy the unique summation property of a geometric series expressed as follows

$$\sum_{n=0}^{N-1} z_k^n = \frac{(z_k)^N - 1}{z_k - 1} = \begin{cases} 1, N = 1 \\ 0, N > 1 \end{cases} \quad \forall k = 0,1,2,..,N-1 \tag{3}$$

Now, we define a periodic, with the period of N, constant modulus, complex discrete-time sequence $e_r(n)$ as

$$e_r(n) \triangleq (z_r)^n = e^{j(2\pi/N)rn} \quad r,n = 0,1,2,... \ , N - 1 \tag{4}$$

This complex sequence over a finite discrete-time interval in a geometric series is expressed according to Eq. (3) as follows [1,2]

$$\frac{1}{N}\sum_{n=0}^{N-1} e_r(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)rn} = \begin{cases} 1, & r = mN \\ 0, & r \neq mN \\ & m,n = \text{integer} \end{cases} \tag{5}$$

This mathematical property is utilized with the factorization into two orthogonal exponential functions where one defines the discrete Fourier transform (DFT) set $\{e_k(n)\}$ satisfying

$$\frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-l)n} = \begin{cases} 1, & k-l = r = mN \\ 0, & k-l = r \neq mN \\ & m,n = \text{integer} \end{cases} \tag{6}$$

The notation (*) represents the complex conjugate function of a function. Note that $\omega_0 = 2\pi / N$ is the n^{th} root of unity on the unit circle and also called the fundamental frequency defined in the unit of radians per cycle. We are going to expand the phase functions in Eq. (6) in order to define GDFT in the following section.

II. GENERALIZED DISCRETE FOURIER TRANSFORM

Let's generalize Eq. (5) by introducing a product function in the phase defined as $\varphi(n) = \varphi_k(n) - \varphi_l(n)$ and expressing a constant amplitude orthogonal set as follows,

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)rn} = \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)\varphi(n)n} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e_k(n) e_l^*(n) \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)[\varphi_k(n)-\varphi_l(n)]n} \qquad (7)\\
&= \begin{Bmatrix} 1, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r = mN \\ 0, & \varphi(n) = \varphi_k(n) - \varphi_l(n) = r \neq mN \\ & m, n = \text{integer} \end{Bmatrix}
\end{aligned}$$

Hence, the basis functions of the new orthogonal set are defined as

$$\begin{aligned}
&e_k(n) \triangleq e^{j(2\pi/N)\varphi_k(n)n} \\
&k \ \& \ n = 0, 1, \ldots, N-1
\end{aligned} \qquad (8)$$

We call this new orthogonal function set of Eq. (8) as the *Generalized Discrete Fourier Transform* (GDFT). It is noted that there are infinitely many function sets with constant power are available.

As an example, one might define the discrete time *rational phase function* $\varphi_k(n)$ in Eq. (8) as the ratio of two polynomials,

$$\varphi_k(n) = \frac{N(n)}{D(n)} = \frac{\sum_{j=1}^{N} a_j n^{b_j}}{\sum_{j=1}^{M} c_j n^{d_j}} \quad N \le M \qquad (9)$$

Let's assume that the denominator polynomial $D(n)$ is equal to one and *the order N* numerator polynomial in *n* is defined as follows

$$\varphi_k(n) = \sum_{j=1}^{N} a_j n^{b_j} = a_1 n^{b_1} + a_2 n^{b_2} + a_3 n^{b3} + \ldots + a_N n^{b_N} \qquad (10)$$

In general, the coefficients $\{a_j\}$ are complex and $\{b_j\}$ are real numbers. Now, we would like to make several remarks that link the proposed GDFT to other known transforms and its potential impact on a multicarrier communications system.

Remark 1: DFT is a special solution of GDFT where $\varphi_k(n) = a_1 = k$ and $a_2 = a_3 = \ldots = a_N = 0$, and $b_1 = b_2 = \ldots = b_N = 0$ in Eq. (9) and Eq. (10) for all k. Note that having constant valued $\{\varphi_k(n)\}$ phase functions makes DFT a linear-phase transform.

Remark 2: In general, k and l parameters do not have to be integer numbers as long as they satisfy the orthogonality conditions of Eq. (7). Since there are N orthogonal functions in the set, one needs to have N distinct and real k values. In that case, the real k values are mapped into k integer numbers from *0* to *N-1* and used as the indices of the basis functions in a set.

Remark 3: There are infinitely many possible GDFT sets available in the phase space with constant power where one can design the optimal basis for the desired figure of merit. If the application considered requires a function set with minimized auto- and cross-correlation properties and does not mind about the non-linear phase, naturally, DFT is not the optimal solution for this scenario. Therefore, one can exploit this fact to design various GDFT's where CDMA and OFDM performances in a multicarrier communications system might be improved over the existing solutions where DFT is used as the transform of choice.

Remark 4: Since DFT is a restricted solution of GDFT, it offers a very limited number of sets to be used in a multicarrier communications system. Therefore, the carrier level security is quite vulnerable for a potential intrusion to the system. In contrast, the proposed GDFT provides many possible carrier sets of various lengths with comparable or better performance than DFT. The availability of rich library of orthogonal constant amplitude transforms with good performance allows us to design adaptive systems where user code allocations are made dynamically to exploit the current channel conditions in order to deliver better performance.

III. GDFT DESIGN METHODS

Let's define the DFT matrix of size NxN as

$$\begin{aligned}
&A_{DFT} = [A_{DFT}(k,n)] = [e^{j(2\pi/N)kn}] \\
&k, n = 0, 1, 2, \ldots, N-1
\end{aligned} \qquad (11)$$

We will design GDFT as the generalization to DFT based on the performance metrics required by the application under consideration. Hence, we can express the GDFT matrix as a product of two matrices as follows

$$
\begin{aligned}
A_{GDFT} &= A_{DFT}\,G \\
A_{GDFT}\,A_{GDFT}^{-1} &= I \\
A_{GDFT}^{-1} &= A_{GDFT}^{*T} \qquad GG^{*T} = I
\end{aligned} \tag{12}
$$

where the notation $(*T)$ indicates that conjugate and transpose operations applied to the matrix.

Note that G is the complex orthogonal generalization matrix yielding A_{GDFT} matrix in Eq. (12) with the desired time and frequency domain features. We are going to introduce several G matrix families that are useful to design A_{GDFT} out of A_{DFT} matrix.

A. *Diagonal G Matrix:*

The diagonal G matrix must be constant amplitude for orthonormality of Eq. (12) and one might define it in the following two forms.

A.1 Constant Valued Diagonal Elements:

The elements of this diagonal matrix are the same constant amplitude complex number as expressed in

$$
G(k,n) = \begin{Bmatrix} e^{j\theta}, & k = n \\ 0, & k \neq n \\ & k,n = 0,1,\ldots N \end{Bmatrix} \tag{13}
$$

This type of G matrix used in Eq. (13) generates θ radians per cycle phase shifted version of A_{DFT} matrix as A_{GDFT}. Moreover, the linear phase property of A_{DFT} is still preserved in this case.

A.2 Non-Constant Diagonal Elements:

The non-zero and non-constant diagonal elements of G matrix are defined as

$$
G(k,n) = \begin{Bmatrix} e^{j\theta_k}, & k = n \\ 0, & k \neq n \\ & k,n = 0,1,\ldots,N \end{Bmatrix} \tag{14}
$$

The rows (basis functions) of A_{GDFT} in Eq. (12) are obtained as the element by element products of A_{DFT} rows with the diagonal non-zero complex sequence of G matrix in this scenario. It is observed that each sample of a given basis function in A_{DFT} is phase shifted independently of the other samples. Therefore, the resulting basis function set is entirely different than DFT function set.

A.3 Non-Constant Two Diagonal Matrices G_1 and G_2

We redefine A_{GDFT} matrix in such a way that phase shaping of basis functions will be even more flexible as shown in the matrix equation

$$
\begin{aligned}
A_{GDFT} &= G_1 A_{DFT} G_2 \\
A_{GDFT} A_{GDFT}^{*T} &= I \\
G_1 G_1^{*T} = I \text{ and } & G_2 G_2^{*T} = I
\end{aligned} \tag{15}
$$

where

$$
G_1(k,n) = \begin{Bmatrix} e^{j\theta_{kk}}, & k = n \\ 0, & k \neq n \\ & k,n = 0,1,\ldots N \end{Bmatrix} \tag{16.a}
$$

and

$$
G_2(k,n) = \begin{Bmatrix} e^{j\gamma_{nn}}, & n = k \\ 0, & n \neq k \\ & k,n = 0,1,\ldots N \end{Bmatrix} \tag{16.b}
$$

Note that the kernel generating A_{GDFT} matrix for this case is expresses as follows

$$
\{e_k(n)\} \triangleq e^{j[(2\pi/N)kn + \theta_{kk} + \gamma_{nn}]} \qquad k,n = 0,1,\ \ldots,N-1 \tag{17}
$$

This design method allows us to uniquely modify the phase of the (k,n) th element of the A_{DFT} matrix as the (k,n) th element of the A_{GDFT} matrix.

B. *Full G Matrix:*

The elements of the orthogonal G matrix in Eq. (12) are not constant amplitude in this case. We design a complex G matrix providing an optimized constant amplitude matrix A_{GDFT} based on a predefined figure of merit related to the application specifications. Now, we define a G matrix

$$G(k,n)=\begin{bmatrix} g_{k,n}e^{j\theta_{k,n}} \\ k,n=0,1,...N \end{bmatrix} \quad (18)$$

Where $g_{k,n}$ and $\theta_{k,n}$ are the amplitude and phase values, respectively, for the $(k,n)^{th}$ element of the matrix. It is noted that $A_{GDFT}^{-1} \neq A_{GDFT}^{*T}$ in Eq. (12) for this case. Hence, it is not orthonormal.

The overall computational cost is the combined implementation of A_{DFT} and G matrices in this version of GDFT design. Since DFT has its efficient fast algorithms, FFT, the complexity of G matrix dictates the required additional computational resources to implement GDFT. Therefore, this point needs to be considered in applications when one generalizes DFT into GDFT.

Remark 5: By inspection, it is observed that popular orthogonal Walsh transforms are special solutions of GDFT.

$$A_{WALSH}=A_{GDFT}=A_{DFT}G$$

Similarly, non-linear phase Walsh-like orthogonal transforms [3], Gold codes [4] and other known binary spreading codes can also be expresses within the GDFT framework.

Remark 6: Oppermann proposed a new family of constant modulus orthogonal spreading codes and also showed in [7] that the well-known Frank-Zadoff and Chu Sequences [9, 10, 11] are the special cases of his family. It is interesting to show below that the Orthogonal Oppermann codes are also special solutions to the GDFT framework.

The Oppermann codes in an *NxN* square matrix notation are defined as [7, 8]

$$A_{OPP}(k,i)=(-1)^{ki}\exp\left(\frac{j\pi(k^m i^p+i^n)}{N}\right) \qquad k,i=1,2,....,N \quad (19)$$

In [8], it was proven that Oppermann codes are orthogonal only for the case of p=1 and m is any positive nonzero integer. Note that if one defines the parameters of Eq. (10) for this case as $a_j=0 \;\; j=3,4,.....,N$ and

$$a_1=\frac{k^m+kN}{2} \qquad b_1=0;\quad a_2=\frac{1}{2};\quad b_2=n-1 \quad (20)$$

then we obtain the equality $A_{GDFT}=A_{OPP}$.

Remark 7: The term "Generalized DFT" was also used by other authors for their methods reported in [12-17] where their focus is of only on linear phase sets. Therefore, the non-linear phase GDFT is the superset of those techniques. Note that linear phase extensions of DFT yield the same auto- and cross correlation performance as the DFT.

IV. GDFT DESIGN EXAMPLE

A. Performance Metrics:

In order to compare performance of spreading code families, we define several objective metrics. All the metrics used in this study depend on *Aperiodic Correlation Functions (ACF)* of spreading code sets. In Eq. (21), the metric $d_{k,i}(m)$ is defined as the ACF between two complex sequences, namely $e_k(n)$ and $e_l(n)$ [18],

$$d_{k,l}(m)=\begin{cases} \frac{1}{N}\sum_{n=0}^{N-1-m} e_k(n)e_l^*(n+m), & 0<m\leq N-1 \\ \frac{1}{N}\sum_{n=0}^{N-1+m} e_k(n-m)e_l^*(n), & 1-N<m\leq 0 \\ 0 \qquad , & |m|\geq N \end{cases} \quad (21)$$

In this paper, four different correlation metrics are used to compare various code families. Namely, a. Maximum Value of Out of Phase Auto-correlation (d_{am}), b. Maximum Value of Cross-correlation (d_{cm}), c. Mean Square Value of Auto-correlation (R_{AC}), and d. Mean Square Value of Cross-correlation (R_{CC}). The definitions of these metrics are as follows [18, 19]

$$d_{am}=\max_{\substack{0\leq k<M \\ 1\leq m<M}}\left\{|d_k(m)|\right\} \quad (22)$$

$$d_{cm}=\max_{\substack{0\leq k,l<M \;\; k\neq l \\ 0\leq m<M}}\left\{|d_{k,l}(m)|\right\} \quad (23)$$

$$d_{\max}=\max\left\{d_{am},d_{cm}\right\} \quad (24)$$

$$R_{AC}=\frac{1}{M}\sum_{k=1}^{M}\sum_{\substack{m=1-N \\ m\neq 0}}^{N-1}|d_{k,k}(m)|^2 \quad (25)$$

$$R_{CC}=\frac{1}{M(M-1)}\sum_{k=1}^{M}\sum_{\substack{l=1 \\ l\neq k}}^{M}\sum_{m=1-N}^{N-1}|d_{k,l}(m)|^2 \quad (26)$$

where M is the set size, and N is the length of each spreading code.

B. Brute Force Search Based Optimal GDFT Codes:

In Eq. (14), G is a complex diagonal matrix with constant modulus elements and we searched the entire phase space with various resolutions using brute force search algorithm in order to find the optimum G matrices. Possible phase values are chosen from the interval $[0, 2\pi]$ with a linear resolution $\Delta\theta$ defined as

$$\Delta\theta = \frac{2\pi}{2^b} \tag{27}$$

where b is the number of bits to represent quantized phase values. Although we have chosen the value of 5 for b in this study, one may increase the phase resolution by choosing larger values of b. The brute force search algorithm is run for 8-length code families based on the performance metrics defined. Their values for optimum code sets along with DFT are tabulated in Table 1.

Table 1: Performance metrics for optimum codes and DFT, N=8.

Optimization Metric	Corresponding Correlation Metrics				
	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
d_{cm}	0.666	0.288	0.666	3.041	0.566
$d_{\max}$	0.366	0.375	0.375	0.818	0.883
R_{AC}	0.125	0.617	0.616	0.088	0.987
R_{CC}	0.875	0.327	0.875	4.375	0.375
DFT N=8	0.875	0.327	0.875	4.375	0.375

C. A Closed Form Phase Shaping Function for GDFT:

The phase function $\varphi_k(n)$ can be decomposed into two functions in the time variable "*n*" as follows

$$\varphi_k(n) = kn + \psi(n) \tag{28}$$

Now, we would like to define a closed form expression for the *phase shaping function (PSF)* $\psi(n)$ in Eq. (28) that is approximating to the brute force based optimal solutions obtained in the previous section via curve fitting. We used the signal processing software tool *Table Curve 2D*, and fitted *PSF* to the phase of optimal complex set obtained based on the performance metric d_{cm} as expressed in

$$\psi(n) = a_1 \exp\left(-(\frac{n-b_1}{c_1})^2\right) + a_2 \exp\left(-(\frac{n-b_2}{c_2})^2\right) \tag{29}$$

This is a second degree *Gaussian function* with six different parameters $\{a_1, b_1, c_1, a_2, b_2, c_2\}$. Our simulation studies showed that the phase function in Eq. (29) provides the minimum value for the performance metric "Maximum value of Cross-correlation, d_{cm}" as well as the minimum value for "Mean Square Value of Auto-correlation, R_{AC}". In Eq. (29), the values of the parameter set, $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ corresponding to the optimal solutions are calculated in order to be able to define closed form *PSFs*. The *PSF* of Eq. (29) provides GDFT solutions of various sizes with good correlation properties. One may obtain infinitely many orthogonal complex spreading code families by changing the set of 6 parameters in Eq. (29) according to the system design specs.

In this paper, we limited our presentation to the optimal search results based on two metrics, namely d_{cm} and R_{AC}. It is observed and shown that the BER performance on AWGN channel with smaller number of users in the system is closely coupled to the d_{cm} parameter. In contrast, the BER performance for multipath fading channel is related to R_{AC}. Our goal is to define the parameters in Eq. (29) generating orthonormal GDFT yielding BER performance improvements for both AWGN and multipath channel models. We employed Genetic Search Algorithm [20] to find optimum *PSFs* for different sizes and design types. The number of initial population, the number of population, the probability of crossover, and the probability of mutation for the search algorithm were chosen as 1000, 100, 0.9 and 0.1, respectively. The phase function given in Eq. (29) is used with different values of its parameters $\{a_1, b_1, c_1, a_2, b_2, c_2\}$ where each one is chosen from the interval of (0, N] with the resolution of 0.25. The algorithm is run for the three different code lengths, N=8, 16, 32. The results are displayed in Tables 2 and 3.

The advantages of the proposed method is its ability of designing a wide selection of orthogonal spreading codes based on the desired performance metrics mimicing the variations of a real world communications channel. Moreover, the proposed technique is an enhancement to the DFT based implementations for performance improvements. Note that the code set with low R_{AC} metric can be used in communication systems where multipath fading is the main concern whereas the set with low d_{cm} values can be used in the systems where multiuser interference is dominant.

Table 2: Various performance metrics when optimal design is based on the metric d_{cm} for code lengths of N = 8, 16, and 32.

Size (N)	Corresponding correlation metrics optimized based on d_{cm} along with DFT				
	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
8 GDFT 8 DFT	0.703 0.875	0.288 0.327	0.703 0.875	3.261 4.375	0.534 0.375
16 GDFT 16 DFT	0.744 0.938	0.248 0.321	0.744 0.938	6.653 9.688	0.557 0.354
32 GDFT 32 DFT	0.794 0.969	0.233 0.319	0.794 0.969	13.68 20.34	0.559 0.344

Table 3: Various performance metrics when optimal design is based on the metric R_{AC} for code lengths of N = 8, 16, and 32.

Size (N)	Corresponding correlation metrics optimized based on R_{AC} along with DFT				
	d_{am}	d_{cm}	$d_{\max}$	R_{AC}	R_{CC}
8 GDFT 8 DFT	0.125 0.875	0.679 0.327	0.679 0.875	0.089 4.375	0.987 0.375
16 GDFT 16 DFT	0.242 0.938	0.700 0.321	0.700 0.938	0.234 9.688	0.984 0.354
32 GDFT 32 DFT	0.604 0.969	0.746 0.319	0.746 0.969	1.165 20.34	0.962 0.344

It is observed that the magnitude of auto-correlation functions of the individual codes in any GDFT set generated from the closed form *PSF* defined in Eq. (29) are exactly the same. In Fig. 1, auto-correlation function (AC) of a code in size 16 GDFT set optimized based on the metric R_{AC} is displayed along with the AC of size 16 DFT set.

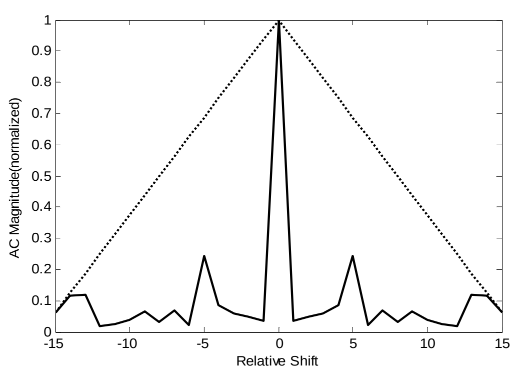

Figure 1: Magnitude of Auto-correlation Functions for low- R_{AC} based GDFT design (solid line) and DFT (dashed line) sets for size N=16.

Similarly, cross-correlation Functions (CC) between the first and second codes of low- d_{cm} based GDFT set and the DFT set are displayed in Fig. 2. These figures highlight the merit of the proposed GDFT framework with non-linear phase.

D. BER on AWGN and Multipath Fading Channels:

In Figure 3, we display BER performance of various GDFT sets designed based on low- d_{cm} for 2 users in an asynchronous AWGN channel along with DFT of size 8.

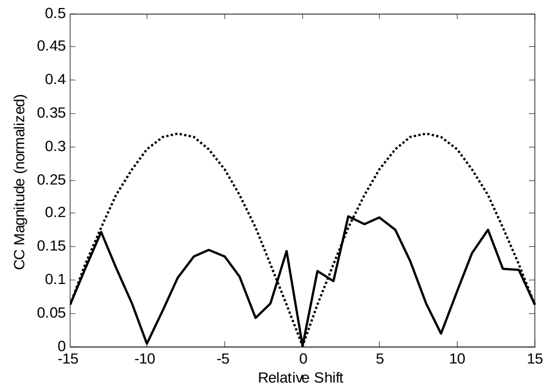

Figure 2: Magnitude of Cross-correlation Functions for low- d_{cm} based GDFT design (solid line) and DFT (dashed line) sets for size N=16.

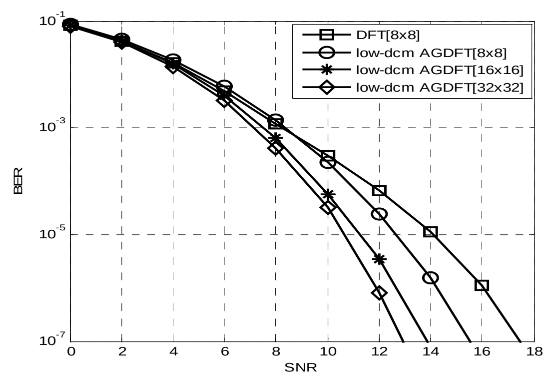

Figure 3: BER Performance of various length GDFTs along with size 8 DFT for 2-users asynchronous AWGN channel.

Fig. 4 displays BER performance of several binary and complex code sets along with GDFT set optimized based on d_{cm} parameter for N=8. Note that in CDMA communications, the multiuser interference (MUI) becomes the dominant factor defining the BER performance as the number of users in the system increases.

We assume a two-ray "multipath channel" with the impulse response of

$$h(t) = \beta_0 \delta(t) + \beta_1 \delta(t - \tau) \qquad (30)$$

In our simulations, the parameters $\{\beta_0, \beta_1\}$ are the Rayleigh distributed random variables defining power of the desired and interfering paths, respectively, and the sum of $E\left[\beta_0^2\right]$ and $E\left[\beta_1^2\right]$ is set to be equal to 1. Fig. 5 displays BER performance of size 8 DFT and AGDFT sets on Rayleigh multipath channel when the power of interfering path is 3dB ($\frac{D}{I}=3dB$) and 5 dB ($\frac{D}{I}=5dB$) less than the power of the desired path and the delay, τ , is set to be equal to T/8. In this example, we used GDFT set generated by *PSF* of Eq. (29) optimized based on minimization of R_{AC} and the resulting R_{AC} values are 4.375 and 0.089 for DFT and GDFT, respectively.

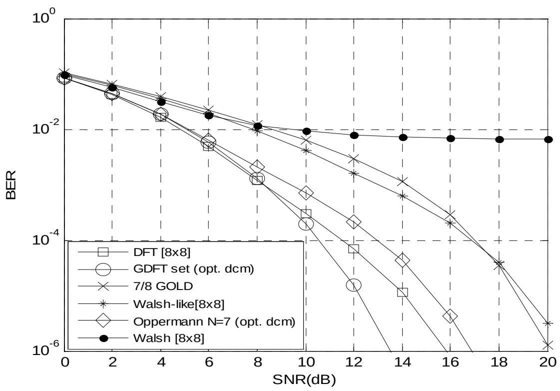

Figure 4: BER performance of various code sets for 2-users and asynchronous AWGN channel with N=8 (N=7 for Gold and Oppermann).

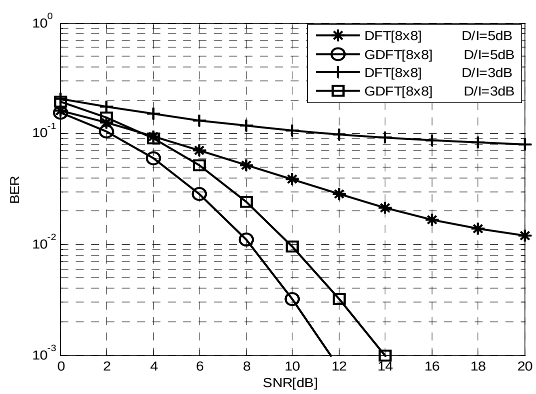

Figure 5: BER Performances of DFT and GDFT spreading code sets over Rayleigh fading channel for 2 users with $D/I=5dB$ and $D/I=3dB$.

From the figure, it is concluded that GDFT set significantly outperforms DFT set on Rayleigh multipath channel due to its low R_{AC} value.

V. CONCLUSIONS

In this paper, we introduced a theoretical framework to design constant modulus transforms efficiently. The proposed GDFT is compared with the industry standard DFT. It is shown that improved correlations of the proposed GDFT technique yield superior BER performance over the known code families in CDMA communications. It is concluded that DFT based engineering solutions including wireless CDMA and OFDM communications might benefit from the proposed GDFT framework.

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