PERFECT RECONSTRUCTION BINOMIAL QMF-WAVELET TRANSFORM

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Abstract

This paper describes a class of orthogonal binomial filters which provide a set of basis functions for a bank of perfect reconstruction Finite Impulse Response Quadrature Mirror Filters (FIR-QMF). These Binomial QMFs are shown to be the same filters as those derived from a discrete orthonormal wavelet approach by Daubechies [13]. The proposed filters can be implemented very efficiently with output scaling, but otherwise no multiply operations. The compaction performance of the proposed signal decomposition technique is computed and shown to be better than that of the DCT for the AR(1) signal models, and also for standard test images.

I. Introduction

Subband coding as an efficient coding technique for signal compression has several considerable attention since its introduction by Crochiere, et.al.[1]. The basic idea is to divide the signal bandwidth into a number of frequency subbands. Each subband is then subsampled and encoded with a bit rate matched to the signal statistics in that subband. These subbands are then reassembled at the receiver. Perfect reconstruction results when the received signal can be reassembled without error except for that introduced by the encoding itself.

The subband technique was first developed for speech compression [2] [3], and then extended to multidimensional signals [4]. Applications to image compression followed, as described in references [5][6].

Perfect reconstruction (PR) Quadrature Mirror Filters (QMF) have been proposed as structures suitable for subband coding [7] [8] [9] [10], and also for multiresolution signal decomposition as might be used in image pyramid coding[11]. More recently, multiresolution signal decomposition methods are being examined from the standpoint of the discrete wavelet transform [12] [13] [14] [15]. In this paper, we describe a class of orthogonal binomial filters which provide basis functions for a perfect reconstruction bank of finite impulse response QMFs. The orthonormal wavelet solutions derived by Daubechies [13] from a discrete wavelet transform approach are shown to be the same as the solutions inherent in the binomial-based filters.

The compaction performance of the Binomial QMF decomposition is computed and shown to be better than the DCT for the Markov source models. The proposed binomial structure is efficient, simple to implement on VLSI, and suitable for multiresolution signal decomposition and coding applications.

II. The Binomial Family

The binomial family of orthogonal sequences [16] is generated by successive differencing of the binomial sequence, which is defined on the finite interval $[0, N]$ by

$$x_0(k) = \begin{cases} \binom{N}{k} = \frac{N!}{(N-k)!k!}, & 0 \le k \le N \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The other members of the binomial family are obtained from

$$x_r(k) = \nabla^r \binom{N-r}{k}, r = 0, 1, \ldots, N \tag{2}$$

where

$$\nabla f(n) = f(n) - f(n-1)$$

is the backward difference operator. Taking successive differences yields

$$\begin{aligned} x_r(k) &= \binom{N}{k} \sum_{\nu=0}^{r} (-2)^\nu \binom{r}{\nu} \frac{k^{(\nu)}}{N^{(\nu)}} \\ &= \binom{N}{k} H_r(k) \qquad\qquad k = 0, 1, \ldots, N \end{aligned} \tag{3}$$

where $k^{(\nu)}$ is the forward factorial function, apolynomial in k of degree ν

$$k^{(\nu)} = \begin{cases} k(k-1)\ldots(k-\nu+1) & , \quad \nu \ge 1 \\ 1 & , \quad \nu = 1 \end{cases} \tag{4}$$

The polynomials, $H_r(k)$, in (4) are the discrete Hermite polynomials, modified versions of which have been used in transform coding [18]. Hence the discrete binomial family $\{x_r(k)\}$ are simply discrete Hermite polynomials windowed by the binomial sequence $\binom{N}{k}$. In this respect, the binomial family is the discrete counterpart to the continuous variable Hermite functions which are Gaussian-windowed Hermite polynomials in the continuous variable domain [19].

The salient properties of the binomial sequences are as follows:

1. These sequences are orthogonal with respect to a weighting function

$$\sum_{k=0}^{N} x_r(k) x_s(k) \binom{N}{k}^{-1} = \binom{N}{r}^{-1} (2)^N \delta_{r-s} \tag{5}$$

2. Transform and Recursion Relation: Starting with the zeroth order binomial, $\binom{N}{k}$, in (1), and the differences in (2) one obtains

$$X_0(z) = Z\left\{\binom{N}{k}\right\} = \sum_{k=0}^{N} \binom{N}{k} z^{-k} = (1 + z^{-1})^N \tag{6}$$

$$\begin{aligned} X_r(z) &= Z\left\{\nabla^r \binom{N-r}{k}\right\} = (1 - z^{-1})^r Z\left\{\binom{N-r}{k}\right\} \\ &= (1 - z^{-1})^r (1 + z^{-1})^{N-r} \end{aligned} \tag{7}$$

This last equation can also be expressed in alternative forms

$$X_r(z) = \left(\frac{1-z^{-1}}{1+z^{-1}}\right) X_{r-1}(z) = \left(\frac{1-z^{-1}}{1+z^{-1}}\right)^r X_0(z) \tag{8}$$

In the time (or spatial) domain, (8) implies the recursive difference equation

$$x_{r+1}(k) = -x_{r+1}(k-1) + x_r(k) - x_r(k-1), \qquad 0 \le k, r \le N \tag{9}$$

with initial values $x_r(-1) = 0$, for $0 \le r \le N$, and initial sequence $x_0(k) = \binom{N}{k}$. Equations (8) and (9) suggest the Binomial Network shown in block diagram form in Fig. 1. The implementation of the binomial family is trivially simple. Since all coefficients are unity, the filter can be realized with just delays and adders – no multipliers are needed. These filters have been used in efficiently processing speech and image signals [17] [20].

3. Time and Frequency Responses of the Binomial Family: The frequency response of the r^{th} member of the binomial family is

$$X_r(e^{j\theta}) = A_r(\theta)e^{j\psi_r(\theta)} \tag{10}$$

where

$$\begin{aligned} A_r(\theta) &= (2^N)(\sin\theta/2)^r(\cos\theta/2)^{N-r} \\ \psi_r(\theta) &= \frac{r\pi}{2} - \frac{N\theta}{2} \end{aligned} \tag{11}$$

4. Quadrature Mirror Filter Properties: The binomial filters are linear phase quadrature mirror filters. From (11), the complementary filters $X_r(z)$, and $X_{N-r}(z)$ have magnitude responses which are mirror images about $\theta = \pi/2$.

$$\left|X_r(e^{j(\frac{\pi}{2}-\theta)}\right| = \left|X_{N-r}(e^{j(\frac{\pi}{2}+\theta)}\right| \tag{12}$$

In time domain, the mirror relation corresponds to

$$x_{N-r}(n) = (-1)^n x_r(n) \tag{13}$$

Additionally, we note that

$$x_r(N-n) = (-1)^r x_r(n) \tag{14}$$

III. Two Channel QMF Bank

We can obtain the conditions for perfect reconstruction from an analysis of the prototype two channel QMF bank shown in Fig. 2. Tracing the signals through the top and bottom branches gives

$$\begin{aligned} \hat{X}(z) &= \frac{1}{2}[H_1(z)K_1(z) + H_2(z)K_2(z)]X(z) \\ &\quad + \frac{1}{2}[H_1(-z)K_1(z) + H_2(-z)K_2(z)]X(-z) \\ &= T(z)X(z) + S(z)X(-z) \end{aligned} \tag{15}$$

Perfect reconstruction requires:

$$\begin{aligned}\text{(i)} \quad S(z) = &\; 0, \text{for all } \; z & (16)\\ \text{(ii)} \quad T(z) = &\; cz^{-n_0}, c \text{ a constant} & (17)\end{aligned}$$

If one chooses

$$\begin{aligned} K_1(z) &= -H_2(-z) \\ K_2(z) &= H_1(-z) \end{aligned} \tag{18}$$

the first requirement is met, $S(z) = 0$, and aliasing is eliminated, leaving us with

$$T(z) = \frac{1}{2}[H_1(-z)H_2(z) - H_1(z)H_2(-z)]$$

next, with N odd, one selects

$$H_2(z) = z^{-N}H_1(-z^{-1}) \tag{19}$$

This choice forces

$$H_2(-z) = -K_1(z)$$

so that

$$T(z) = \frac{1}{2}z^{-N}[H_1(z)H_1(z^{-1}) + H_1(-z)H_1(-z^{-1})] \tag{20}$$

Therefore, the perfect reconstruction requirement reduces to finding an $H(z) = H_1(z)$ such that

$$\begin{aligned} Q(z) &= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = \text{constant} \\ &= R(z) + R(-z) \end{aligned} \tag{21}$$

This selection implies that all four filters are causal whenever $H_1(z)$ is causal.

The PR requirement can be recast in time domain form by noting that $R(z)$ is a spectral density function, whose inverse is the autocorrelation sequence

$$\begin{aligned} \rho(n) &= \sum_{k=0}^{N} h(k)h(k+n) = \rho(-n) \\ &\stackrel{\text{def}}{=} h(n) \odot h(n) \end{aligned} \tag{22}$$

where $\odot$ indicates a correlation operation. It can be shown that [22] Eq. (21) is satisfied when

$$\rho(2n) = \sum_{k=0}^{N} h(k)h(k+2n) = 0, \qquad n \neq 0 \tag{23}$$

IV. The Binomial QMF

It is now a straight forward matter to impose PR condition of (??)on the binomial family. First, we take as the half-band low-pass filter

$$h(n) = \sum_{r=0}^{\frac{N-1}{2}} \theta_r x_r(n)$$

or

$$H(z) = \sum_{r=0}^{\frac{N-1}{2}} \theta_r (1+z^{-1})^{N-r}(1-z^{-1})^r \tag{24}$$

The corresponding autocorrelation sequence for $h(n)$ is found to be

$$\rho(n) = \sum_{n=0}^{\frac{N-1}{2}} \theta_r^2 \rho_{rr}(n) + 2\sum_{l=1}^{\frac{N-3}{2}} \sum_{\nu=0}^{\frac{N-1}{2}-2l} \theta_\nu \theta_{\nu+2l} \rho_{\nu,\nu+2l}(n) \tag{25}$$

$$\rho_{rs}(n) = \sum_{k=0}^{N} x_r(k) x_s(n+k) \longleftrightarrow R_{rs}(z) \tag{26}$$

where $\rho_{rs}(n)$ is the cross correlation of $x_r(n)$ and $x_s(n)$, Finally, the PR requirement is

$$\rho(n) = 0, \qquad n = 2,4,\ldots,N-1 \tag{27}$$

This condition gives a set of $\frac{N-1}{2}$ nonlinear algebraic equations, in the $\frac{N-1}{2}$ unknowns $\theta_1, \theta_2 \ldots, \theta_{\frac{N-1}{2}}$.

The values of θ_r, for $N = 3,5,7$, (corresponding to 4,6,8 tap filters respectively) are given in Table 1 (where $\theta_0 = 1$). As seen, there are more than one filter solutions for a given N. For example, with N =3, one obtains $\theta_1 = \sqrt{3}$, and also $\theta_1 = -\sqrt{3}$. The positive θ_1 corresponds to a minimum phase solution, while the negative θ_1 provides a non-minimum phase filter. The magnitude responses of both filters are identical, although in our derivation, no linear phase constraint on $h(n)$ was imposed; it is noteworthy, that the phase responses are almost linear, the non-minimum phase filters even more so.

Table 2 provides the normalized 4,6,8 tap filter coefficients, $h(n)$ for both minimum and non-minimum phase cases. The implementation of these half-band filters is trivially simple. The θ_r weight is applied to the corresponding r^{th} tap point in the Binomial Network for $r = 1,\ldots,\frac{N-1}{2}$. These are the only multiplications needed when using the Binomial Network as the half-band QMF structure rather than the $h(n)$ weights directly. This binomial structure can lead to even simpler filter algorithms wherein the θ_r weights can be absorbed in the quantizers. For $N = 3$, there is a multiplier-free structure.

Remarks: These Binomial QMFs satisfy all the conditions of an orthonormal wavelet basis with regularity. It is noteworthy that the solutions for all even-tapped filters are exactly the same as those derived in [13] from a wavelet approach.

V. Performance of Binomial QMF-Wavelet Transform

The performance of the new signal decomposition scheme is compared with the industry standard, the Discrete Cosine Transform (DCT) in this section.

The energy compaction power of any unitary transform is a commonly used performance criterion. The gain of transform coding over PCM is defined as [21]

$$G_{TC} = \frac{\frac{1}{M}\sum_{k=0}^{M-1} \sigma_k^2}{\left[\prod_{k=0}^{M-1} \sigma_k^2\right]^{1/M}} \tag{28}$$

where σ_k^2 are transform coefficient variances. This measure assumes that all coefficients have the same probability density function.

Similarly the gain of subband coding over PCM is defined as

$$G_{SBC} = \frac{\frac{1}{M}\sum_{l=0}^{M-1} \sigma_l^2}{\left[\prod_{l=0}^{M-1} \sigma_l^2\right]^{1/M}} \tag{29}$$

Here σ_l^2 is the variance of the signal in the l^{th} subband. This formula holds for a two-band split in a regular tree structure.

We assume a Markov 1 source model with autocorrelation

$$R(k) = \rho^{|k|}, \qquad k = 0, \pm 1, \ldots \tag{30}$$

and calculate G_{TC} and G_{SBC} for different cases. These results are displayed in Table 3. Equations (28) and (29) are easily extended to the 2D case for separable transforms and separable QMFs.

The results demonstrate that the 6-tap Binomial QMF outperforms the comparable sized DCT in both theoretical performance as well as for standard test images. We conclude therefore that the 6-tap Binomial QMF provides a better alternative to the DCT for image coding.

VI. Conclusions

An efficient perfect reconstruction binomial QMF-Wavelet signal decomposition structure is proposed. The new technique utilizes the binomial network which has only addition operations. This technique provides a set of filter solutions with very good amplitude responses and band split. The phase responses of these filters are linear-like. Non minimum phase solutions provide even better linear-like phase characteristics. These filters are the same as the orthonormal Wavelet solutions proposed by Daubechies [13].

The Binomial QMF-Wavelet signal decomposition structures have better compaction than the industry standard DCT for Markov sources and real images. The new structure is a very powerful technique. The 6-tap filters provide better performance than the DCT (8×8) for image decomposition. These QMF-filters have a very simple algorithm to implement on VLSI and are a good competitor to the DCT for signal decomposition and coding applications.

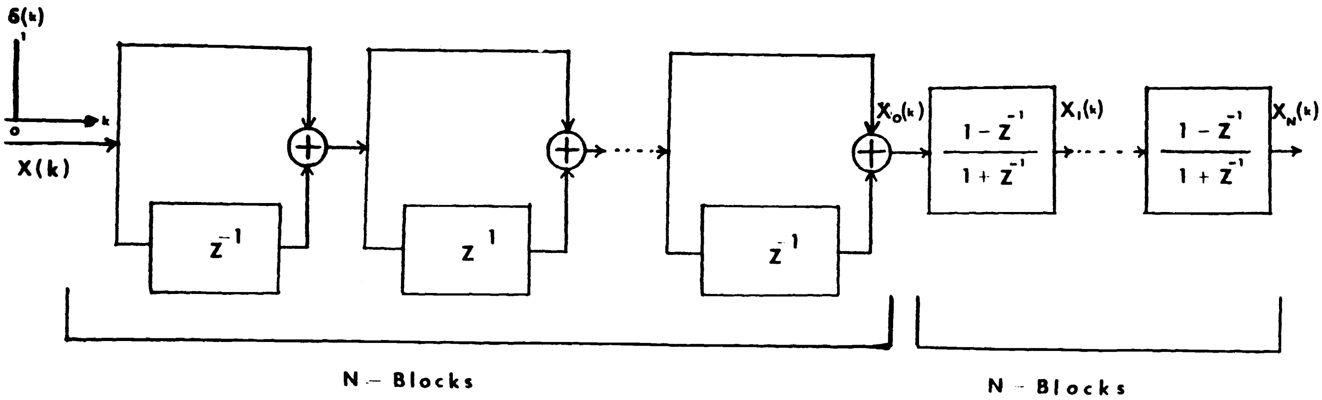

Figure 1: Binomial Network

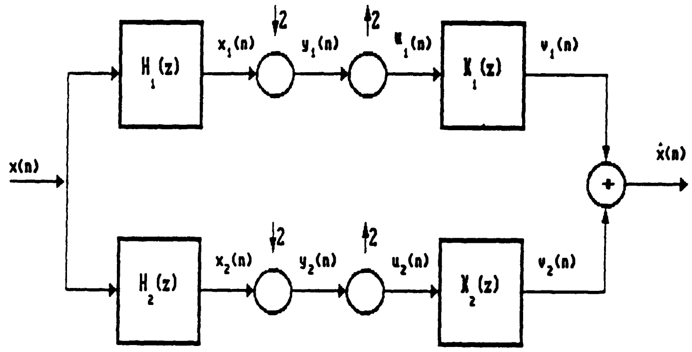

Figure 2: Two Channel QMF Bank

N=3		
θ_r	set 1	set 2
θ_0	1	1
θ_1	$\sqrt{3}$	$-\sqrt{3}$

N=5		
θ_r	set 1	set 2
θ_0	1	1
θ_1	$\sqrt{2\sqrt{10}+5}$	$-\sqrt{2\sqrt{10}+5}$
θ_2	$\sqrt{10}$	$\sqrt{10}$

N=7				
θ_r	set 1	set 2	set 3	set 4
θ_0	1	1	1	1
θ_1	4.9892	-4.9892	1.0290	-1.0290
θ_2	8.9461	8.9461	-2.9705	-2.9705
θ_3	5.9160	-5.9160	-5.9160	5.9160

Table 1: θ_r values for $N = 3, 5, 7$

n	h(n)			
	Mini Phase	Non-Minimum Phase		
	4 tap	4 tap		
0	0.48296291314453	-0.1294095225512		
1	0.83651630373780	0.2241438680420		
2	0.22414386804201	0.8365163037378		
3	-0.12940952255126	0.4829629131445		
	6 tap	6 tap		
0	0.33267055439701	0.0352262935542		
1	0.80689151040469	-0.0854412721235		
2	0.45987749838630	-0.1350110232992		
3	-0.13501102329922	0.4598774983863		
4	-0.08544127212359	0.8068915104046		
5	0.03522629355424	0.3326705543970		
	8 tap	8 tap	8 tap	8 tap
0	0.23037781098452	-0.0105973984294	-0.0757657137833	0.0322230981272
1	0.71484656725691	0.0328830189591	-0.0296355292117	-0.0126039690937
2	0.63088077185926	0.0308413834495	0.4976186593836	-0.0992195317257
3	-0.02798376387108	-0.1870348133969	0.8037387521124	0.2978578127957
4	-0.18703481339693	-0.0279837638710	0.2978578127957	0.8037387521124
5	0.03084138344957	0.6308807718592	-0.0992195317257	0.4976186593836
6	0.03288301895913	0.7148465672569	-0.0126039690937	-0.0296355292117
7	-0.01059739842942	0.2303778109845	0.0322230981272	-0.0757657137833

Table 2: Binomial QMF-Wavelet filters, $h(n)$, for $N = 3, 5, 7$

	ρ	G_{TC}	4-tap	6-tap	8-tap	16-tap
4 × 4 Trans.	0.95	5.71	6.43	6.77	6.91	7.08
or	0.85	2.59	2.82	2.95	3.01	3.07
	0.75	1.84	1.95	2.02	2.05	2.09
4-band QMF	0.65	1.49	1.56	1.60	1.62	1.64
(2 levels)	0.5	1.23	1.26	1.28	1.29	1.30
8 × 8 Trans.	0.95	7.63	8.01	8.53	8.74	8.99
or	0.85	3.03	3.11	3.27	3.34	3.42
	0.75	2.03	2.06	2.14	2.17	2.22
8-band QMF	0.65	1.59	1.60	1.65	1.67	1.69
(3 level)	0.5	1.27	1.28	1.30	1.31	1.32

Table 3: Compaction Comparison; DCT vs Binomial-QMF for several AR(1) sources.

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