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FILTER BANKS AND WAVELETS IN SIGNAL PROCESSING: A CRITICAL REVIEW

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ABSTRACT

This paper provides an overview of subband and wavelet theories. It emphasizes their strong relations. The practical merits of these decomposition techniques in signal processing are examined. The current status of this active research field is summarized and it concludes with the discussion of potential extentions for future study.

1. INTRODUCTION

Orthogonal transforms, particularly non-overlapping or block transforms, have been widely used in signal processing applications. Lately, the discrete cosine transform(DCT) has become the standard signal decomposition tool in still image and video coding algorithms, e.g. JPEG, H.261, MPEG[1-2]. On the other hand the subband decomposition has been proposed in the literature for better spectral decomposition of speech and image-video signals[3-15]. The subband decomposition employs a set of filters or overlapping basis which covers the whole spectrum. As the uncertainty principle states longer the time function implies better frequency localization. The block transforms are mostly defined in time domain orthogonality conditions while the subbands emphasize the frequency domain behavior of filter functions in the basis. It is interesting that these once competing signal decomposition techniques are presently considered as the members of the unified linear transform theory covering both overlapping and non-overlapping bases. More recently, the degrees of freedom in subband decomposition were emphasized and optimal filter basis design was proposed[16]. Additionally, the duality of time

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and frequency domains is considered for the joint time-frequency localized basis designs[17-18]. The subband theory has evolved to the most general cases of K-dimensional, M-band unequal bandwidths[19-20].

Wavelet transforms have been proposed recently as a new multiresolution decomposition tool for continuous-time signals. It was shown by Daubechies[21] that the two-band orthonormal PR-QMF coefficients serve as the inter-scale coefficients in wavelet theory[22]. It was also shown that the dyadic-tree of subband technique serves as fast wavelet transform algorithm. The linkages of wavelet and subband theories have stimulated interdisciplinary research activities by mathematicians and engineers on this topic. The wavelet theory has quite matured and its meritful applications in analog signal processing are still to come. This paper reviews the subband and wavelet theories. It emphasizes their linkages and examines their practical merits in signal processing.

2. ORTHONORMALITY IN NON-OVERLAPPING AND OVERLAPPING TRANSFORM BASES

Consider a set of N orthogonal sequences $\{h_r(k)\}$, r = 0,1,...,N-1, on the interval $[0, N-1]$ which satisfy the condition

$$\sum_{n=0}^{N-1} h_r(n)h_s^*(n) = \delta_{r-s} \tag{1}$$

The transform basis defined in Eq.(1) consists of N sequences with durations N in time. These transforms are called unitary or block(non-overlapping) transforms in the literature. Typical examples of block transforms are discrete cosine transform(DCT), discrete Fourier transform(DFT), discrete Walsh-Hadamard transform(DWHT). The optimal block transform is the Karhunen-Loeve transform(KLT) which provides uncorrelated transform coefficients for a given input covariance[23]. The time and frequency functions of DCT basis for $N = 8$ are displayed in Figure 1. It is seen from Fig.1 that the frequency characteristics of basis functions are poor. Better frequency behavior requires longer duration basis functions. The increase in the durations of basis functions causes the overlapping which must be handled properly in orthonormality conditions. The extension of unitary block transforms to subbands is called the paraunitary solution[8][11]. Figure 2 displays an M-band paraunitary filter bank, i.e. $\hat{f}(n) = f(n - n_0)$, which satisfies the following conditions[14]:

a. The analysis and synthesis functions have the same time duration L and related as

$$G_r(z) = z^{-(L-1)} H_r(z^{-1})$$

b. Each filter function is normalized as

$$\sum_{n=0}^{L-1} h_r^2(n) = 1$$

c. Ecah filter is orthogonal to its own shifts by M and its multiples,

$$\sum_k h_r(k) h_r(Mn + k) = 0, \qquad n \neq 0$$

d. Filter $h_r(k)$ is orthogonal to $h_s(k)$ and its shifts as

$$\sum_k h_r(k) h_s(Mn + k) = 0$$

Remarks:

(**1**) Notice the additional tail orthogonality properties in subbands compared to block transforms.

(**2**) In subband solutions there are more degrees of freedom compared to block transforms. This lends itself for optimal filter bank solutions[16].

(**3**) Computationally efficient subband transform algorithms are also possible similar to efficient block transform algorithms[24].

2.A Two-band Orthonormal PR-QMF and Hierarchical Subband Trees

The two-band orthonormal PR-QMF solutions are the special case of M-band with $M = 2$ given in the previous section[5][7][14]. The two-band PR-QMF is used repetitively in a hierarchical subband tree to obtain a flexible decomposition of signal spectrum. Fig. 3 displays the four-band dyadic filter bank and its spectral split as an example of hierarchy. Different types of subband trees are suggested in the literature[25].

Remarks:
(**4**) The dyadic(octave-band) subband tree serves as fast wavelet transform algorithm with proper initialization as discussed in Section 4.
(**5**) The time and frequency behaviors of subband trees should be monitored carefully. The original design is for the purpose of two-band split. It is not guaranteed that the product filters are proper for the task they are assigned for[18].

3. WAVELET TRANSFORMS

The wavelet transforms have recently been proposed as a new mathematical tool for multiresolution decomposition of continuous-variable signals. The wavelet transform has more flexible time-frequency localization properties than the short-time Fourier transform(STFT)[21][14][26][27].

The wavelet transform is defined in terms of dilation and translation of a prototype kernel, $\psi(t)$. It maps a function $f(t)$ on a time-scale space (a, b) by

$$W_f(a,b) = \int_{-\infty}^{\infty} \psi_{ab}(t) f(t) dt \triangleq < \psi_{ab}, f > \tag{2}$$

This transform is invertible provided $\psi(t)$ satisfies the admissibility[21]; then

$$f(t) = \frac{1}{C} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{dadb}{a^2} W_f(a,b) \psi_{ab}(t) \tag{3}$$

$$C = \int_0^{\infty} \frac{|\Psi(\Omega)|^2 d\Omega}{\Omega} \tag{4}$$

Hence $\Psi(0)$ must be zero, so that $\psi(t)$ behaves as the impulse response of a band-pass analog filter.

3.A Discrete Wavelet Transform(DWT)

The scaling and translation parameters are discretized in DWT as $a = 2^m$, $b = n2^m$. This results in the wavelet family

$$\psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m}t - n) \tag{5}$$

The discrete wavelet transform expansion is given as

$$d_{m,n} = \int_{-\infty}^{\infty} f(t) \psi_{m,n}(t) dt \triangleq < f(t), \psi_{m,n}(t) > \tag{6}$$

$$f(t)=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty} d_{m,n}\psi_{m,n}(t) \quad (7)$$

3.B Construction of Wavelets From Orthonormal PR-QMFs

The orthonormality conditions of wavelet/scaling bases and their construction from orthonormal PR-QMFs are summarized as[21]:

a. The band-pass wavelets are orthonormal in intra-scale as well as in inter-scale,

$$<\psi_{m,n}(t),\psi_{m',n'}(t)>=\delta_{m-m'}\delta_{n-n'} \quad (8)$$

b. The complementary low-pass scaling function $\phi(t)$ is orthonormal only in intra-scale as

$$<\phi_{mn}(t),\phi_{mn'}(t)>=\delta_{n-n'} \quad (9)$$

where

$$\phi_{mn}(t)=2^{-m/2}\phi(2^{-m}t-n) \quad (10)$$

c. Complementary property of wavelet and scaling bases is

$$<\psi_{mn(t)},\phi_{m'n'}(t)>=0 \quad (11)$$

d. Let $h_0(n)$ and $h_1(n)$ be the low and high-pass filters, respectively, of a two-band orthonormal filter bank with the added property that $H_0(e^{j\omega})=0$ at $\omega=\pi$. Then it can be shown that $\phi(t)$ and $\psi(t)$ are constructed from the discrete-time filters or equivalently from inter-scale coefficients, via the fundamental wavelet equations of the containment property as[21]

$$\begin{aligned}\phi(t) &= \sum_n h_0(n)\phi(2t-n)\leftrightarrow\Phi(\Omega)=\prod_{k=1}^{\infty}H_0(e^{j\omega/2^k})\\ \psi(t) &= \sum_n h_1(n)\phi(2t-n)\leftrightarrow\Psi(\Omega)=H_1(e^{j\omega/2})\prod_{k=2}^{\infty}H_0(e^{j\omega/2^k}) \quad (12)\end{aligned}$$

It turns out that the orthonormality and finite support of the scaling and wavelet functions are set up by the orthonormality and finite duration of discrete-time filters $h_0(n)$ and $h_1(n)$. This equation shows how to construct orthonormal wavelet bases of compact support by using orthonormal two-band PR-QMFs.

Remark:

(**6**) The construction of orthonormal, compact support wavelet bases starts with the design of finite duration orthonormal two-band PR-QMFs.

4. WAVELET EXPANSION AND DYADIC SUBBAND TREE

Let us represent $f(t)$ by using the orthonormal scaling functions at full resolution

$$f(t) = \sum_n c(0,n)\phi(t-n) = \sum c(0,n)\phi_{0n}(t) \tag{13}$$

where the scaling coefficients

$$c(0,n) = < f(t), \phi_{0n}(t) > \tag{14}$$

Let us assume that the sequence $c(0,n)$ is the input to the discrete-time dyadic subband tree as shown in Fig.3. The sequences $h_0(n)$ and $h_1(n)$ are orthonormal PR-QMFs which generated $\Phi(\Omega)$ and $\Psi(\Omega)$ in Eq.(12).

By the containment property, $f(t)$ can be decomposed into a lower resolution approximation by the scaling basis and the approximation error is represented by the complementary wavelet basis as[21]

$$f(t) = \sum_n d(1,n)\psi_{1n}(t) + \sum_n c(1,n)\phi_{1n}(t)$$

The coarse approximation can be approximated by a coarser approximation using the corresponding scaling basis and this new approximation error is also represented by the wavelet basis of lower resolution as

$$f(t) = \sum_n d(1,n)\psi_{1n}(t) + \sum_n d(2,n)\psi_{2n}(t) + \sum_n c(2,n)\phi_{2n}(t)$$

Therefore, $f(t)$ can be represented as a low-pass approximation at the $1/L$-resolution plus sum of L detail wavelet components at the finer successive resolutions. The coefficients in this multiresolution expansion are the outputs of the orthonormal dyadic subband tree. Hence, it is called *fast wavelet transform.*

Remarks:

(**7**) Orthonormal dyadic subband tree serves as a fast wavelet transform algorithm if it is driven by the scaling coefficients of the top level, $c(0,n)$.

(**8**) If the data is already sampled it is hard to justify to perform wavelet transform on it. The proper way to decompose the discrete-time signals is to employ discrete-time decomposition techniques like block or subband transforms.

5. DISCUSSIONS AND CONCLUSIONS

The theories of subband and wavelet decompositions have matured. Some of the signal processing applications already use subband decomposition. For example, several audio compression algorithms like MPEG are using subband technique and their commercial products are in the market. The block transforms, which are special subband filter banks, have been widely used in many signal processing and coding applications. The DCT is the standard decomposition tool in image-video coding algorithms, e.g. H.261, JPEG, MPEG. The improved and efficient subband schemes with multiresolution feature are still a good candidate for next generation image-video processing and coding standard algorithms.

The subband approach has available degrees of freedom to be used for tuning the basis to the application at hand. This point has been recognized lately and expected to be utilized in future applications. Additionally, the structure or type of the spectral split needs more attention for improvements in signal decomposition problems. More flexible, unequal bandwidth subband filter banks adapted to the input spectrum are expected to provide improvements in future applications. Since the theory of subbands is well understood, we should expect more work on applications specific problems and engineering of subband bases. This also implies more concern about the algorithmic issues in the designs. The spectral analysis and spectral decomposition have been two overlapping fields in signal processing. We expect that these two subjects of signal processing will merge even further for the improvements in applications.

The linkages of wavelet and subband theories have stimulated the recent inter-disciplinary activities in these topics. The mathematical elegance of wavelet theory has created high expectations from this decomposition tool for signal processing applications. But it is still to come. The term "wavelet" has been liberally used in the literature even for discrete-time signal processing. The current trend in the field is to reassess the real merits of wavelet transforms, in other words analog filter banks in signal processing terminology, and search for new applications of it.

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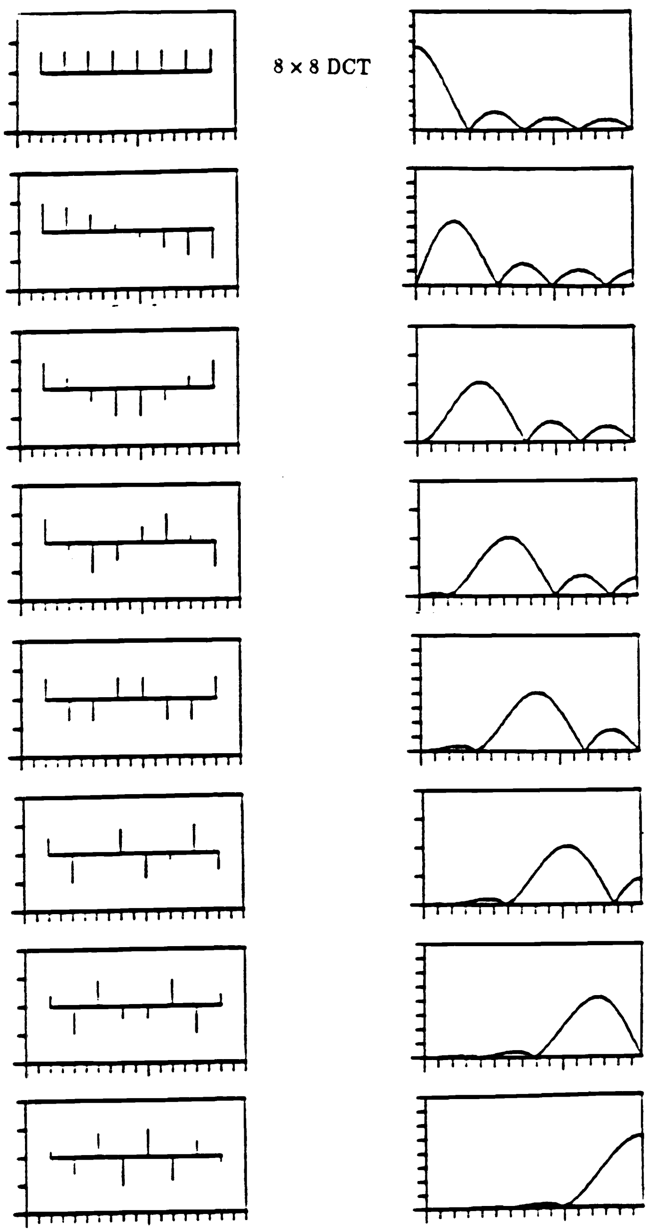


Figure 1: The basis functions of 8x8 DCT in time and frequency domains.

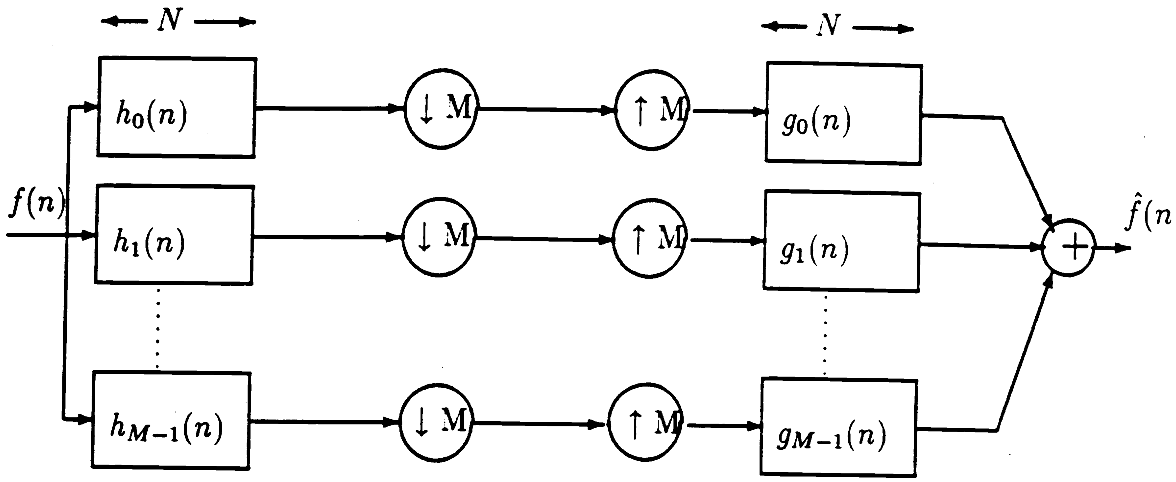
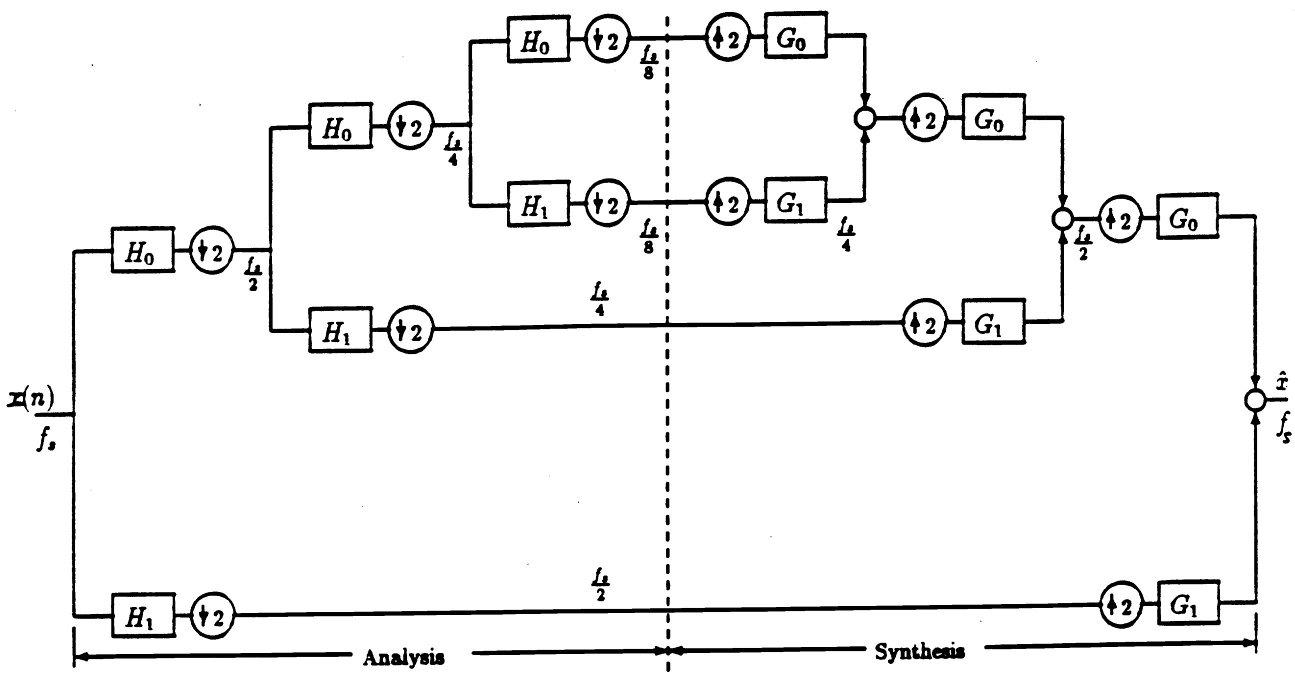
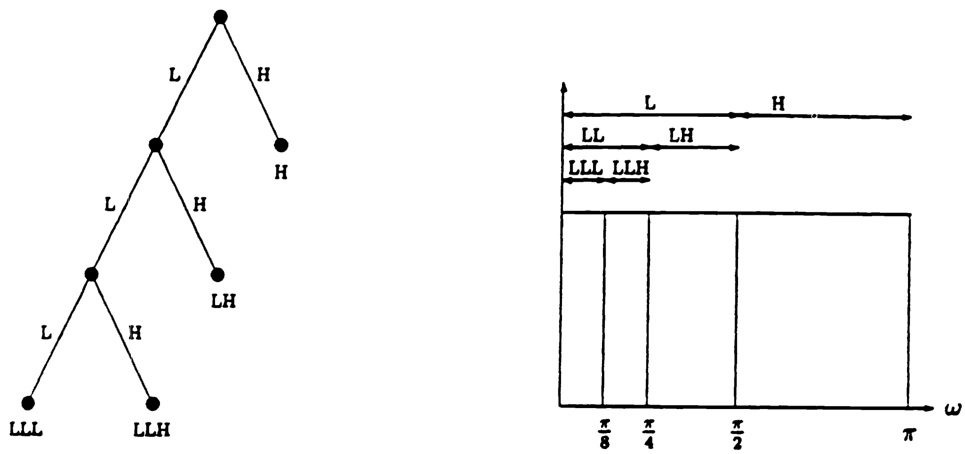


Figure 2: M-band paraunitary filter bank.



a)



b)

Figure 3: a) Four-band, three-level dyadic filter bank, b) the tree and its frequency split using ideal two-band PR-QMF.