STATISTICALLY OPTIMIZED PR-QMF DESIGN

Hakan Caglar, Yipeng Liu and Ali N. Akansu

New Jersey Institute of Technology
Department of Electrical and Computer Engineering
Center for Communications and Signal Processing Research
University Heights, Newark, NJ 07102

ABSTRACT

A multivariable optimization problem is set to design 2-band PR-QMFs in this paper. The energy compaction, aliasing energy, step response, zero-mean high-pass filter, uncorrelated subband signals, constrained nonlinearity of the phase-response, and the given input statistics are simultaneously considered in the proposed optimal filter design technique. A set of optimal PR-QMF solutions and their optimization criteria along with their energy compaction performance are given for comparison. This approach of PR-QMF design leads to an input driven adaptive subband filter bank structure. It is expected that these optimal filters outperform the well-known fixed PR-QMFs in the literature for image and video coding applications.

I. INTRODUCTION

2-band Perfect Reconstruction Quadrature Mirror Filters(PR-QMF) have become popular multiresolution signal decomposition tools particularly for signal coding applications[1][2][3][4]. Their modular nature leads to the hierarchical subband trees which have been widely used in the literature. Additionally, it is shown that the 2-band PR-QMFs are the crucial components of the orthonormal wavelet basis design procedure[5][6].

This paper deals with the optimal 2-band PR-QMF design problem. The approach taken here considers a set of design variables which are of great practical interest in image coding. Some of these variables have been considered in the filter design field earlier but this study considers all of them simultaneously to obtain the optimal solutions.

Section II. introduces the variables of optimization problem and their practical significance. Section III. discusses the mathematical preliminaries and lays the ground for the objective functions of the optimization. Sections IV. and V. looks into two different cases of optimal PR-QMF design problem. Section VI. discusses the future research and concludes the paper.

II. VARIABLES OF OPTIMIZATION AND THEIR PRACTICAL SIGNIFICANCE

The proposed statistically optimized PR-QMF design technique considers several parameters of practical significance in the filter design. These parameters, namely energy compaction, aliasing, step response, zero mean high-pass filter, uncorrelated band signals, constrained non-linear phase response, and given input statistics characteristics are combined to define the objective function of

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the optimization problem. Some of these features have been well known in the filter design field and used by several researchers in the literature. Johnston has considered the aliasing minimization in his QMF design procedure[7]. But as known, his filters are not PR. Additionally, it is a classical approach to filter design to minimize the stop band energy. This study intuitively benefited from the earlier work in the field but also has significant improvements in the solution of PR-QMF problems. We included the following variables in the design of PR-QMFs:

a) Orthonormal PR Requirement: This set of requirements is included in the design to obtain the perfect reconstruction condition which is of interest here. The PR becomes important particularly in signal coding applications.

b) Energy Compaction: It is a well known, desired characteristic from any orthogonal signal decomposition technique which has direct connections with the rate-distortion theory. The significance and derivation of this measure is given in [8]. This measure has been widely used for performance comparisons of different signal decomposition techniques in the literature[4].

c) Aliasing Energy: Any PR signal decomposition technique satisfies the conditions of alias cancelation. But in practice, since all the decomposition bands or coefficents are not used, or the different levels of subband quantization noise leads to the aliasing problem. Its significance has been noticed particularly in image coding applications in the literature. It is known that the aliasing problem causes annoying patterns in encoded images at low bit-rates.

d) Step Response: The representation of edges in an image is a crucial problem. The edge structures are localized in time therefore they should be represented by the time-localized basis functions. Otherwise the well known ringing artifacts occur in encoded images. Therefore the step responses of the filters in the filter banks should be considered during the design procedure[9]. It is a well known phenomenon called uncertainty principle which states that a signal can not be localized perfectly in one domain without the worst concentration in the other[10]. This is easily observed in an extreme case that if an edge assumed as a unit sample function in time, then the best representing basis function for this signal is also the unit sample function which has the best localization in time. On the other hand we know the frequency localization of this function is the worst. Similarly the best localized frequency functions, e.g. ideal band-pass functions, have infinite durations in time.

It has been reported that the human visual system is able to resolve the time frequency domains therefore a joint time-frequency localization should be considered in a practically meritful filter bank design. This trade-off between the time and frequency resolutions reflects itself implicitly into the aliasing and step response characteristics of the designed filters.

e) Zero Mean High-Pass: Most of the practical signal sources have their significant energy around the DC frequency. Therefore any practically useful signal decomposition technique should be able to represent the DC frequency component within only one basis function. Following this argument one should constrain the high-pass QMF function of having a zero mean.

f) Uncorrelatedness of Subband Signals: It is a well-known fact in signal coding field that any good signal representation technique should be able to provide uncorrelated transform coefficients or subband signals. The well known Karhunen-Loeve Transform(KLT) is a typical example of this characteristic in the block transforms. Similarly the filter bank solutions under the constraints of this feature are sought in this study.

It is also noteworthy to mention that the uncorrelatedness and the maximum energy compaction requirements merge in the KLT solutions of block transforms.

g) Constrained Non-linear Phase Response: Since there can not be any linear-phase orthonormal PR-QMF solution, the linearity of the phase response of the basis functions are relaxed. Linear phase and PR are two conflicting conditions in orthonormal 2-band QMF design. But it is also known that severe phase nonlinearities create undesired degradations in image and video applications. Therefore a measure which indicates the level of nonlinearity of the filter phase response is included as a parameter in this optimized filter design.

h) Given Input Statistics: The characteristics of the input spectral density function are very important for the design variables explained earlier. Therefore the whole optimization procedure is related to the given input statistics. This also leads to the input adaptive solutions which may be useful in some of the applications for the non-stationary sources.

III. MATHEMATICAL PRELIMINARIES

The variables of the optimization which have been introduced in the previous section will be defined here. The objective function of the optimization problem will be set for the optimal solutions.

a) Orthonormal PR Requirement: The high-pass filter is assumed to be the mirror of the half-band low-pass filter $\{h(n)\}$ of length $2N$ which is also expressed in the vector form $\underline{h}$. Hence the orthonomality condition can easily be written as

$$\underline{h}^T \underline{h} = 1 \tag{1}$$

The perfect reconstruction conditions of orthonormal 2-band PR-QMF is derived as

$$\sum_n h(n)h(n+2k) = \delta(k) \tag{2}$$

We can now express Eqs.(1) and (2) in the matrix form

$$\underline{h}^T C_i \underline{h} = 0 \qquad i = 1, 2, ..., N-1 \tag{3}$$

where C_i are the corresponding shuffling matrices as

$$C_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ . & . & & & & . \\ . & . & & & & . \\ 0 & 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad ,\dots, \quad C_{N-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ . & . & & & . & . \\ . & . & & & . & . \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{4}$$

b) Energy Compaction: The output energy of the low pass filter $h(n)$ for the given zero-mean input covariance matrix R_{xx} can be written as

$$\sigma_L^2 = \underline{h}^T R_{xx} \underline{h} \tag{5}$$

We are now looking for the solution which maximizes Eq.(5). It is clear that this will be the sufficient condition to maximize the gain of transform coding over PCM(G_{TC}) which is given for the two-band case as

$$G_{TC} = \frac{\sigma_x^2}{(\sigma_L^2 \sigma_H^2)^{1/2}} = \frac{\sigma_x^2}{\sigma_L \sigma_H} \tag{6}$$

where the input signal variance is expressed in the unitary case as

$$\sigma_x^2 = \frac{1}{2}(\sigma_L^2 + \sigma_H^2)$$

c) Aliasing Energy: The aliasing energy component for the low-pass filter output of 2-band PR-QMF bank can be written for the given input spectral density function$S_{xx}(e^{j\omega})$ as

$$\sigma_A^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{-j\omega})|^2 S_{xx}(e^{j(\omega+\pi)}) |H(e^{j(\omega+\pi)})|^2 d\omega \tag{7}$$

The corresponding time-domain version of this relation can be rewritten as

$$\sigma_A^2 = \sum_k [\rho(n) * (-1)^n \rho(n)] R_{xx}(k) \tag{8}$$

where $\rho(n)$ is the autocorrelation sequence of $h(n)$ defined as

$$\rho(n) = h(n) * h(-n)$$

and $R_{xx}(k)$ is the autocorrelation sequence of the input. The optimal solution searched should minimize the aliasing energy component of the low-pass filter output as given in Eq.(8) in 2-band PR-QMF case.

Differently from the earlier design procedures in the literature, the aliasing energy is related to the spectral density function of interest rather than the deviations of the designed filter's frequency characterictics from the ideal filter.

d) Step Response: The unit step response of the filter $h(n)$ can be written as

$$a(n) = h(n) * u(n)$$

where $u(n)$ is the unit step sequence. The error energy between the unit step response $a(n)$ and the unit step sequence $u(n)$ can be expressed as

$$E_s = \sum_{k=0}^{2N-1} [\sum_{n=0}^{k} h(n) - 1]^2 \tag{9}$$

E_s is minimized for the optimal solution. The optimization variable E_s does not consider the symmetry of the unit step response around the step point. As known the ringing problem of image coding may be caused by an overshoot or undershoot. But this point is addressed later that this symmetry condition of the unit step response is directly related to the linear phase condition of the desired filter.

e) Zero Mean High-Pass Condition: It is important that the high pass filter has zero mean as disscussed earlier. This means that

$$\sum_n (-1)^n h(n) = 0 \tag{10}$$

This requirement implies that there should be at least one zero of the low pass filter $h(n)$ at $\omega = \pi$.

f) Uncorrelatedness of Subbands: Since this is a desired feature particularly for signal coding applications it is considered as one of the variables of the optimal filter design problem. This characteristic is defined as

$$E\{y_L(n)y_H(n)\} = R_{LH}(0) = \sum_n [\sum_l h(l)(-1)^l h(n-l)] R_{xx}(n) = 0 \qquad \text{for all } m \tag{11}$$

g) Constrained Non-linear Phase Response: Since the severe nonlinearity of the phase response in filter banks is a cause of degradation in visual signal processing applications the constrained phase nonlinearity which does not cause any practical problem should be considered in the optimal filter design. Nonlinearity measure of the phase response is related to the non-symmetry of the unit sample response and defined as

$$E_p = \sum_n [h(n) - h(2N-1-n)]^2 \tag{12}$$

h) Given Input Statistics: The optimal solutions searched here are related to the input signal statistics. This leads to the input driven adaptive filter solutions which may be employed to deal with nonstationarities of the real world signal sources. This study assumes an autoregresive, order 1, $AR(1)$ source model with the correlation coefficient $\rho = 0.95$ which is a crude approximation to still images. The correlation sequence of this source is expressed as

$$R_{xx}(m) = \rho^{|m|} \quad m = 0, \pm 1, \pm 2, \dots \tag{13}$$

From Eq.(13) the corresponding teoplitz covariance matrix R_{xx} is easily obtained.

IV. OPTIMAL PR-QMF DESIGN BASED ON ENERGY COMPACTION

This optimization problem consists of the variables of a), b) in the previous section, for an $AR(1)$ source of g) from the previous section with $\rho = 0.95$.

We can now set the objective function J which is to be maximized as

$$\max\{J\} = \underline{h}^T R_{xx} \underline{h} + \lambda_0[1 - \underline{h}^T \underline{h}] + \lambda_1[\underline{h}^T C_1 \underline{h}] + \dots + \lambda_i[\underline{h}^T C_i \underline{h}] \tag{14}$$

Hence,

$$\frac{\partial J}{\partial \underline{h}} = 0$$

therefore

$$R_{xx}\underline{h} + \lambda_1 C_1 \underline{h} + \dots + \lambda_i C_i \underline{h} = \lambda_0 \underline{h} \tag{15}$$

If the terms in the left side of the equation are combined as

$$R\underline{h} = \lambda_0 \underline{h} \tag{16}$$

where

$$R = R_{xx} + \lambda_1 C_1 + ... + \lambda_i C_i$$

Eq.(16) looks like a classical eigenvalue problem but here the matrix R has unknown parameters in it. The vector $\underline{h}$ which satisfies Eq.(16) is the optimal PR-QMF half-band low-pass filter. Tables 1 and 2 provide the 4, 6, and 8 tap filter coefficents obtained from this optimization procedure.

V. OPTIMAL PR-QMF DESIGN BASED ON EXTENDED SET OF VARIABLES

The aliasing energy, step response, and constrained non-linear phase characteristics additional to the constraints of Eq.(14) and zero-mean high-pass filter condition are included in the objective function in this section. The objective function of this optimization problem is easily set as

$$\max\{J\} = \underline{h}^T R_{xx} \underline{h} - \alpha \sum_k [\rho(n) * (-1)^n \rho(n)] R_{xx}(k) - \beta \sum_{k=0}^{2N-1} [\sum_{n=0}^{k} h(n) - 1]^2 - \gamma \sum_n [h(n) - h(2N-1-n)]^2 \tag{17}$$

with the set of unitary, PR, and zero-mean constraints

$$\begin{aligned} \sum_n h(n)h(n+2k) &= \delta(k) \\ \sum_n (-1)^n h(n) &= 0 \end{aligned} \tag{18}$$

Tables 3a, b, and c display three different sets of 4,6,8 tap optimal PR-QMF solutions for the indicated parameter sets $\{\alpha, \beta, \gamma\}$ and the 2-band filter bank energy compaction performance of those filters for $AR(1)$ source model with $\rho = 0.95$.

VI. FUTURE RESEARCH AND CONCLUSIONS

We have developed a frame to design statistically optimized 2-band PR-QMFs. This procedure considers the effects of the uncertainty principle of the time-frequency signal analysis. This is implicitly succeeded by considering the effects of the aliasing energy and the unit step response of the designed PR-QMFs for the given statistics.

The proposed solutions succeed to consider several practical problems of signal coding in the optimization problem. Therefore there are quite a few different filter solutions available under this approach. All the PR-QMF solutions presented here are based on $AR(1)$ source model with $\rho = 0.95$. Therefore better tuned statistical models of image sources should be included in the optimization problem for further studies.

This approach somehow leads to the solutions of the input driven adaptive filter banks to overcome the difficulties of varying source characteristics in the non-stationary cases. It is expected that the statistically optimized PR-QMFs introduced in this paper perform better than the the well-known PR-QMFs in the literature[6][7]. These filters should be incorporated in image, video processing and coding applications to prove their merits.

It should also be emphasized that some of the characteristics considered in the proposed optimized PR-QMF design procedure may not be significant in the 2-band case. But if a tree structure based on 2-band PR-QMFs are created it is clearly observed that all of these characteristics become important. Therefore they should be considered simultaneously.

This approach can be extended to the $M-$band PR filter bank problem but as expected the procedure in that case is computationally quite complex for the larger values of M.

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Table 1. Filter coefficients of 4, 6, and 8 tap optimal PR-QMFs based on Eq.(14) and their 2-band energy compaction for AR(1) source with $\rho = 0.95$.

n	h(n)	h(n)	h(n)
0	0.3228416	0.3856583	0.4884862
1	0.7492217	0.7962807	0.8322184
2	0.5311527	0.4281477	0.2261980
3	-0.0612652	-0.1408510	-0.1327707
4	-0.2055118	-0.1066993	
5	0.0426724	0.0516771	
6	0.0616565		
7	-0.0265680		
G_{TC}	3.8567	3.7972	3.6529

Table 2. Optimal PR-QMFs having zero-mean high-pass additonal to Eq.(14) and their 2-band energy compaction for AR(1) source with $\rho = 0.95$.

n	h(n)	h(n)	h(n)
0	0.3179736	0.3856632	0.4829630
1	0.7488967	0.7962818	0.8365163
2	0.5349441	0.4281412	0.2241439
3	-0.0588333	-0.1408519	-0.1294095
4	-0.2058194	-0.1066976	
5	0.0425224	0.0516769	
6	0.0600086		
7	-0.0254790		
G_{TC}	3.8567	3.7972	3.6426

Tables 3 a,b,c Three sets of optimal PR-QMFs obtained from Eqs.(17) and (18) with different values of α, β, and γ parameters and their 2-band energy compaction for AR(1) source with $\rho = 0.95$

$\alpha = 1,\ \beta = 0.01,\ \gamma = 0.01$			
n	h(n)	h(n)	h(n)
0	0.3806122	0.4502598	0.4760834
1	0.6764663	0.8101271	0.8383073
2	0.4606891	0.3343153	0.2310233
3	-0.0829168	-0.1460764	-0.1312005
4	-0.1999896	-0.0774683	
5	0.0562299	0.0430560	
6	0.0657951		
7	-0.0326725		
G_{TC}	3.8397	3.7768	3.6422

$\alpha = 1,\ \beta = 0.01,\ \gamma = 0.1$			
n	h(n)	h(n)	h(n)
0	0.3656289	0.3880495	0.4502767
1	0.6456302	0.8249238	0.8441088
2	0.4881103	0.3785860	0.2568301
3	-0.0478489	-0.1458196	-0.1370020
4	-0.2378422	-0.0595286	
5	0.0540503	0.0280026	
6	0.0912098		
7	-0.0447258		
G_{TC}	3.8150	3.7498	3.6356

$\alpha = 2,\ \beta = 0.01,\ \gamma = 1$			
n	h(n)	h(n)	h(n)
0	0.0213971	0.0110556	0.0032908
1	-0.0983231	-0.0809281	0.7103669
2	0.0234699	-0.0155682	0.7038160
3	0.7096470	0.6908207	-0.0032601
4	0.6889528	0.7116194	
5	0.1015962	0.0972142	
6	-0.0267929		
7	-0.0058133		
G_{TC}	3.6608	3.5351	3.2026