Wavelets and Filter Banks

A Signal Processing Perspective

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Spectral analysis and signal decomposition continue to find wide use in a multitude of engineering disciplines. The basic idea in signal decomposition is to separate the spectrum into its constituent subspectral components and then process them individually, based on the application at hand. An analysis/synthesis system performs the inverse function, composing the original spectrum back by using its subspectral components. If the reconstructed spectrum perfectly matches the original one, the analysis/synthesis structure is called perfect reconstruction (PR). When the analysis and synthesis function sets are the same, the transform is called orthonormal, in contrast to the bi-orthogonal transform, which has different analysis and synthesis bases. Both of these transforms satisfy perfect reconstruction, although they may have different properties.

Wavelets are a relative newcomer to signal decomposition, and offer a flexible analog transform to provide multiresolution signal decomposition [1-2]. Their advantages over Fourier and short-time Fourier transforms (STFTs) are significant, and the linkages and practical commonalities of these two transform techniques have generated interdisciplinary research activities among mathematicians, physicists and electrical engineers [3-7]. This article presents the fundamentals of wavelet transform theory. In this context, the subject's mathematical rigor is avoided. We discuss the differences between the conventional STFT and wavelet transforms from a time-frequency "tiling" point of view. Then, we highlight the significant role of discrete-time filter banks in wavelet theory, and assess the practicality of wavelets in signal processing applications.

Fourier and Short-Time Fourier Transforms

Fourier analysis decomposes a signal into its frequency components and determines their relative strengths. We define the Fourier Transform as:

$$F(\Omega) = \int_{-\infty}^{\infty} f(t) e^{-j\Omega t} dt \longleftrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\Omega) e^{j\Omega t} d\Omega \tag{1}$$

Equation 1 indicates that this transform is not able to represent any time-local properties of the signal, $f(t)$. It washes out any time-local variations of the signal.

The short-time Fourier transform (STFT) positions a window function $g(t)$ at τ on the time axis, and calculates the Fourier transform of the windowed signal as

$$F(\Omega,\tau) = \int_{-\infty}^{\infty} f(t)\, g^*(t-\tau)\, e^{-j\Omega t} dt \tag{2}$$

When the window $g(t)$ is a Gaussian function, the STFT is called a Gabor transform. The basis functions of this transform are generated by modulation and translation of the window function $g(t)$, where Ω and τ are modulation and translation parameters, respectively. The fixed time window $g(t)$ is the limitation of STFT since it causes a fixed time-frequency resolution. This is explained by the uncertainty principle for the transform pair $g(t) \leftrightarrow G(\Omega)$ [8]:

$$\sigma_T \; \sigma_\Omega \geq \frac{1}{2} \tag{3}$$

where σ_T and σ_Ω are the root mean square (RMS) spread of $g(t)$, and $G(\Omega)$:

$$\sigma_T^2 = \frac{\int t^2 |g(t)|^2 dt}{\int |g(t)|^2 dt} \qquad \sigma_\Omega^2 = \frac{\int \Omega^2 |G(\Omega)|^2 d\Omega}{\int |G(\Omega)|^2 d\Omega} \tag{4}$$

When τ increases, the window function translates in time. On the other hand, the increase in Ω causes a translation in frequency with a constant bandwidth. Therefore, the resolution cell $\sigma_T\sigma_\Omega$ in the time-frequency plane is constant for any frequency Ω and time shift τ, as shown by the "tiles" of fixed area and shape in Fig. 1.

Wavelet Transform

The scaling property of a Fourier transform for the time-frequency function pair $\psi(t) \leftrightarrow \Psi(\Omega)$ is expressed as

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$$\frac{1}{\sqrt{a}}\psi\left(\frac{t}{a}\right) \longleftrightarrow \sqrt{a}\,\Psi(a\Omega) \quad (5)$$

where $a > 0$ is a continuous scale or resolution parameter. Equation 5 shows that a contraction in one domain causes a dilation in the other. This provides the foundation for a nonuniform "tiling" of the time-frequency plane utilized in wavelet transform theory.

The wavelet family is defined by scale and shift parameters a, b as in [1]:

$$\psi_{ab}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right) \quad (6)$$

where the transform kernel $\psi(t)$ is a zero-mean band-pass function. The wavelet transform of a given function $f(t)$ is defined as

$$W(a,b) = \int_{-\infty}^{\infty} \psi_{ab}(t)\; f^*(t)dt = \langle \psi_{ab}, f \rangle \quad (7)$$

where $a \in R^+$, $b \in R$ are the scale and shift parameters, respectively. The notation * denotes the complex conjugate.

For large values of a, the basis function becomes a dilated version of the prototype wavelet, while for small a this function is a contracted version of the wavelet function; respectively, they cause lower and higher frequency bands in the frequency domain. Hence, the scaling parameter a provides the mathematical means for a flexible "tiling" of the time-frequency plane by the wavelet transform. This is a consequence of the scaling property of the Fourier transform, as given in Eq. 5. Figure 2 displays the time-frequency plane showing resolution cells for the wavelet transform. The fundamental differences between the STFT and the wavelet transform are visualized in Figs. 1 and 2. The wavelet transform provides a set of differently shaped time-frequency tiles, while the shapes of the STFT tiles are fixed. For decomposition, this is a desired feature—a more efficient representation of signals.

The wavelet transform is called continuous if the scaling and translation parameters, a and b, respectively, are continuous [1]. The continuous wavelet transform has two drawbacks: redundancy and impracticality. These problems are solved by discretizing the transform parameters (a,b). Let the parameter sampling lattice be:

$$a = a_o^m \quad b = n\, b_o a_o^m$$

so that

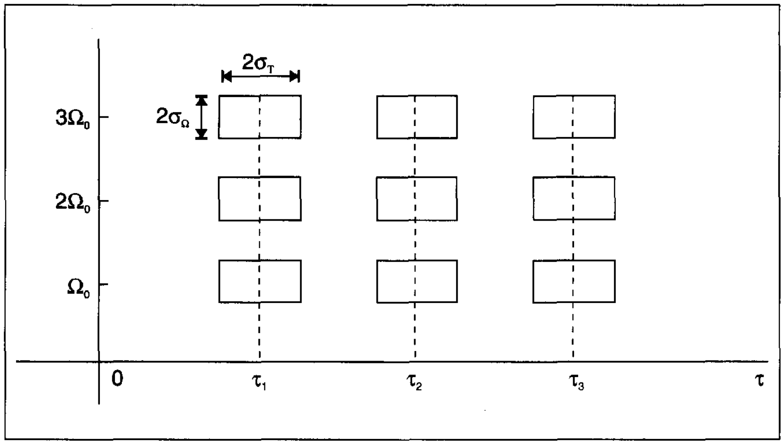
$$\psi_{mn}(t) = a_o^{-m/2}\psi(a_o^{-m}t - nb_o) \quad (8)$$

where m, n $\in$ Z. If this set is complete in $L^2(R)$ for some choice of $\psi(t)$, a, b, then the basis functions $\{\psi_{mn}\}$ are called *affine* wavelets. Hence, we can express any f(t) $\in L^2$ (R) as the superposition

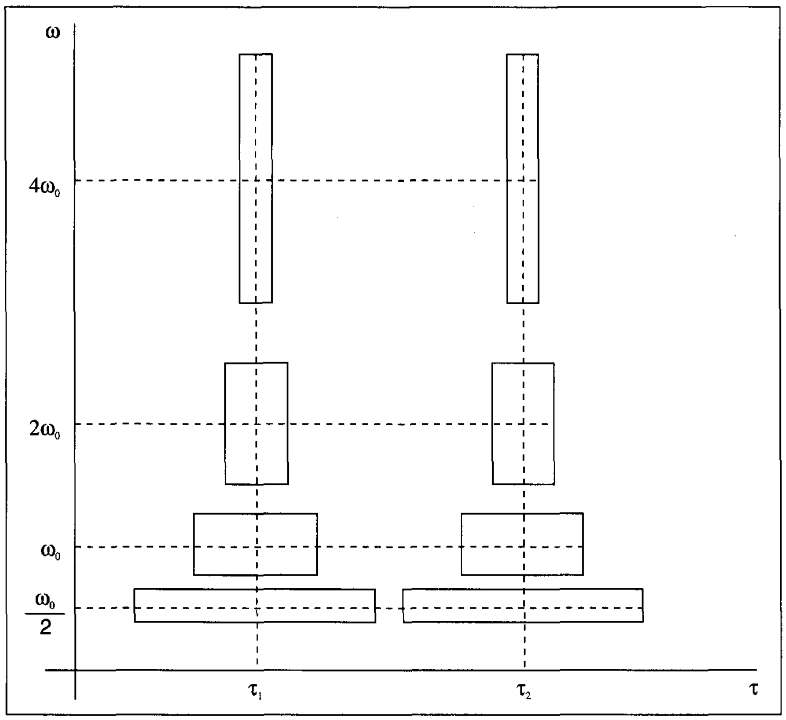
$$f(t) = \sum_m \sum_n d_{m,n}\psi_{mn}(t) \quad (9)$$

where the wavelet coefficient $d_{m,n}$ is the inner product

$$d_{m,n} = \langle f(t), \psi_{mn}(t) \rangle = \frac{1}{a_o^{m/2}}\int f(t)\psi(a_o^{-m}t - nb_o)dt \quad (10)$$



1. Time-Frequency plane showing resolution cells for STFT.



2. Time-Frequency plane showing resolution cells for wavelet transform.

The wavelets are band-pass functions with zero DC-components and generated from the transform kernel or mother wavelet $\{\psi(t)\}$. As seen from Eq. 9, the pure wavelet expansion requires an infinite number of resolutions for the complete representation of the signal. On the other hand, $f(t)$ can be represented as a low-pass approximation at scale L plus the sum of L detail (wavelet) components at different resolutions. The latter form is a more practical representation and points out the complementary role of the scaling basis in such representations. This finite resolution wavelet representation is expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} c_{L,n} a_o^{-L/2} \phi\left(\frac{t}{a_o^L} - nb_o\right) + \sum_{m=1}^{L} \sum_{n=-\infty}^{\infty} d_{m,n} a_o^{-m/2} \psi\left(\frac{t}{a_o^m} - nb_o\right) \quad (11)$$

where the scaling coefficient is the inner product

$$c_{L,n} = \langle f(t), \phi_{Ln}(t) \rangle = \frac{1}{a_o^{L/2}} \int f(t) \phi(a_o^{-L} t - n\, b_o) dt \quad (12)$$

Figure 3 displays the Daubechies (6-tap) wavelet and scaling functions in time and frequency domains. The band-pass characteristics of the wavelet function along with the low-pass nature of the scaling function are observed in Fig. 3.

The complementary scaling basis $\{\phi(t)\}$ in multiresolution wavelet analysis has the containment property within the two adjacent resolutions as:

$$\phi(t) = 2 \sum_n h_o(n)\, \phi(2t - n) \quad (13)$$

The sequence $\{h_o(n)\}$ in this equation consists of so-called interscale basis coefficients. It is interesting that $\{h_o(n)\}$ is identical to the unit sample response of a low-pass filter in a two-band paraunitary filter bank for the case of orthonormal wavelet bases. The linkage of discrete-time filter banks and wavelet transforms will be emphasized in the next section.

Similarly, the band-pass wavelet function can be expressed as a linear combination of translates of the scaling function of the adjacent resolution, $\phi(2t)$:

$$\psi(t) = 2 \sum_n h_1(n)\, \phi(2t - n) \quad (14)$$

This is the fundamental wavelet equation. The expansion coefficients $\{h_1(n)\}$ in this equation will be similarly identified with the high-pass filter of the two-band filter bank structure given in Fig. 4a.

Discrete Wavelet Bases and Filter Banks

The discrete-time filter banks have been well studied and widely used in the signal processing field [4][7][9]. This signal analysis/synthesis tool has found most of its applications in speech processing and coding, image-video processing and coding, and machine vision. The two-band orthonormal (paraunitary) filter bank and orthonormal wavelet theory have strong links. More specifically, the dyadic subband tree structure serves as the fast wavelet transform algorithm if the proper initialization at the top resolution level is performed.

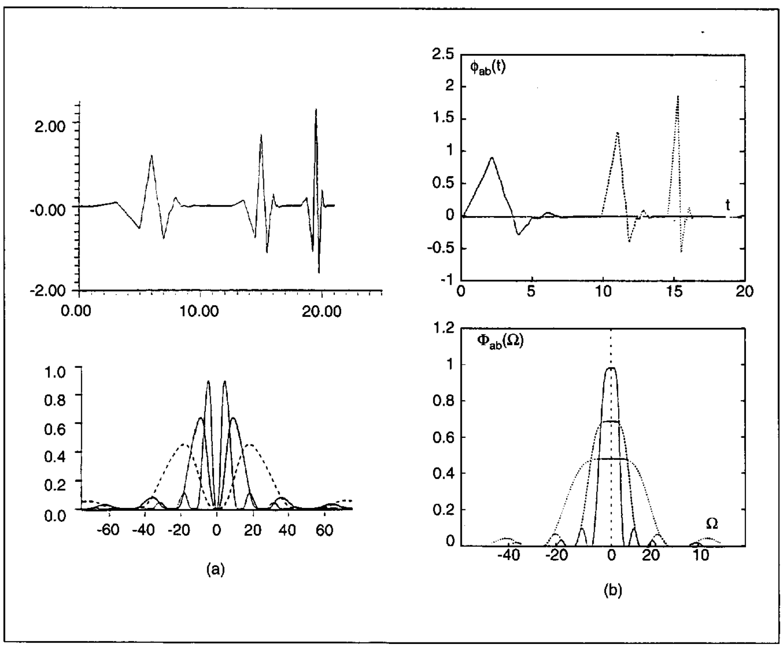
Otherwise, some error in the wavelet representation is evident [4]. The elegance of the orthonormal wavelet theory developed by Daubechies is in its illustration of the linkages of the discrete-time subband filter banks and continuous time wavelet bases. This linkage allows the design of continuous-time wavelet bases via the design of their discrete-time counterparts; namely, the equal bandwidth, two-channel, perfect reconstruction quadrature mirror filter (PR-QMF) banks. The continuous and discrete-time variables meet in the limit. This linkage adds significant flexibility to the design of wavelet transform bases. A generic two-band PR-QMF bank structure is given in Fig. 4a. The detailed treatment of filter banks and their interrelations with wavelet transforms can be found in [4].

The orthonormality conditions of wavelet and scaling bases along with their relations to the discrete-time filter banks will be summarized next without derivations.

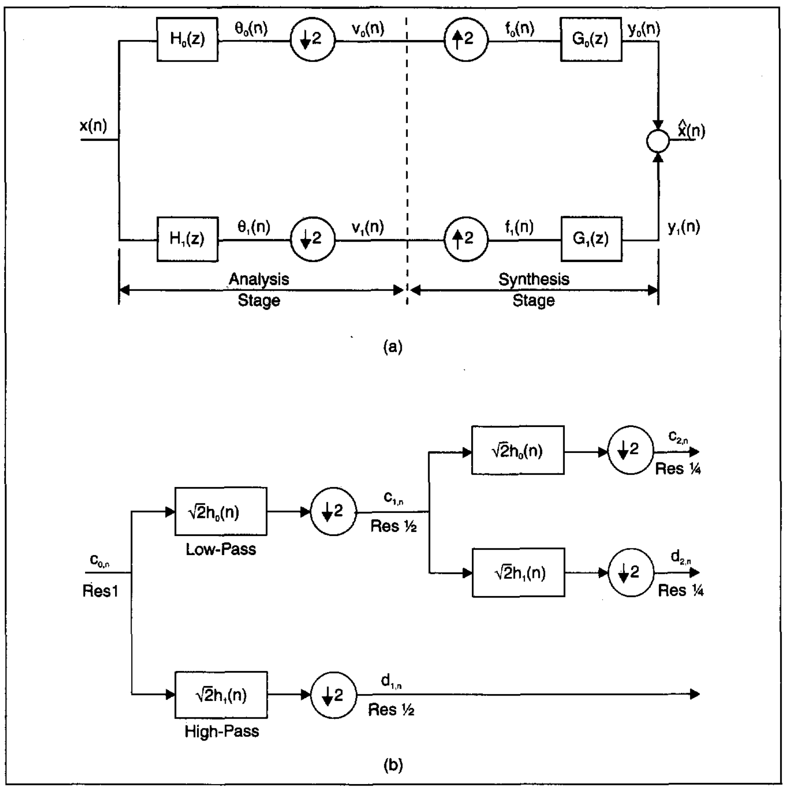
(1) The wavelets are orthonormal in *intra-* and inter-scales, as:

$$\int \psi_{mn}(t) \psi_{m',n'}(t) dt = \delta_{m-m'} \delta_{n-n'} \quad (15)$$

(2) The complementary scaling function of wavelet theory has only *intra-scale orthonormality, as*



3. (a) Daubechies (6-tap) wavelet function and its dilations in time and frequency domains; (b) complementing scaling function and its dilations in time and frequency.



4. (a) Two-band filter bank structure; (b) a two-level dyadic (octave-band) subband tree structure used for a fast wavelet transform with $a_0 = 2$, $b_0 = 1$.

$$\int \phi_{mn}(t)\, \phi_{mn'}\, dt = \delta_{n-n'} \tag{16}$$

where

$$\phi_{mn}(t) = 2^{-m/2}\, \phi(2^{-m}t - n)$$

(3) The complementary property of the wavelet and scaling bases is given as

$$\int \psi_{mn}(t)\phi_{m'n'}(t)dt = 0 \tag{17}$$

for all m, n, m', and n'.

(4) Let $h_0(n)$, $h_1(n)$ constitute a two-band discrete-time PR-QMF bank with the added property of $H_0(e^{j\omega}) = 0$ at $\omega = \pi$. Then, it can be shown that the scaling function $\phi(t)$ and its complementary wavelet kernel $\psi(t)$ are constructed from the discrete-time filters or the inter-scale coefficients, via the fundamental wavelet equations of the containment property, as in [1]:

$$\phi(t) = \sum_n h_o(n)\, \phi(2t-n) \longleftrightarrow \Phi(\Omega) = \prod_{k=1}^{\infty} H_o(e^{j\omega/2^k})$$

$$\psi(t) = \sum_n h_1(n)\phi(2t-n) \longleftrightarrow \Psi(\Omega) = H_1(e^{j\omega/2}) \prod_{k=2}^{\infty} H_o(e^{j\omega/2^k}) \tag{18}$$

It turns out that the orthonormality and finite support of the scaling and wavelet functions are satisfied by the orthonormality and finite duration properties of filters $h_0(n)$ and $h_1(n)$. Equation 18 provides the rules for the construction of wavelet and scaling bases. The wavelet basis design, which is analog, starts with the design of a two-band discrete-time paraunitary filter bank. The Fourier transforms of the wavelet and scaling functions are then obtained by the infinite products of filter functions.

The role of filter sequences $\{h_0(n)\}$ and $\{h_1(n)\}$ in two-band PR-QMF banks and fast wavelet transform algorithms are seen in Figure 4. Those two operations will be identical if and only if $c_{0,n} = x(n)$, where $c_{0,n}$ are the scaling coefficients of full resolution and are defined as

$$c_{0,n} = \int x(t)\, \phi(t-n)dt$$

This is valid only for the ideal *Sinc* function, which has an infinite duration in time. In practice, $c_{0,n}$ is not equal to $x(n)$. Therefore, a wavelet transform in discrete-time implies a degree of imperfectness. In contrast, the subband filter bank in Fig. 4a is the right choice since the signal or function to be transformed is already sampled, hence discrete-time.

Advantages and Observations

In general, the wavelet transform provides a mathematical tool for flexible "tiling" of the time-frequency plane. It is a multiresolution transform of continuous variable. Both the continuous and discrete wavelet transforms are continuous variable transforms. The term "discrete" comes from the discretization of scaling and translation parameters of the wavelet transform [1]. In addition, the wavelet and scaling bases of different resolutions may be considered analog filter banks similar to discrete-time filter banks [4].

Recent wavelet research has provided very little novelty, if any, for discrete-time signal processing applications. It is natural to employ the block and subband transforms in discrete-time signal processing. These techniques have been extensively studied in the literature for the last two decades. Their theories are well established and applications are being developed. The block transforms, particularly discrete Fourier transform (DFT) and discrete cosine transform (DCT), have been successfully utilized in many signal processing applications. The latter is the decomposition technique of the current international standard image and video compression algorithms, e.g., JPEG, H.261, MPEG. Lately, the subband transforms have been used in diverse areas such as digital audio broadcasting, video and image coding, speech coding, spread spectrum communications, and others.

The theory of wavelet transforms is very elegant. Unfortunately, however, most of its signal processing applications reported in the literature have been ill-fitted to the task. Future studies in the signal processing applications of wavelet transforms should emphasize the continuous variable or analog nature of these transforms. This is the most meaningful path, which checks the very foundations of wavelet theory. We expect that in the future, the wavelet transform will find its merit in analog filter banks and signal processing.

Conclusion

The wavelet transform has strong links with discrete-time filter banks, which have been well applied on sampled data. Most of the applications in the wavelet literature are based on sampled data, and are mistakingly attributed to wavelet theory; however, these operations are actually dyadic subband decomposition operations. Thus far, there have been only a few applications to analog signals that fit properly with the fundamentals of wavelet theory. The practical merits of the elegant wavelet transform in signal processing are still to be explored for analog applications.

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References

1. I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," *Comm. in Pure and Applied Math.*, **41**, pp. 909-996, 1988.

2. S. Mallat, "A Theory for Multiresolution Signal Decomposition: the Wavelet Representation," *IEEE Trans. on Pattern Analysis and Mach. Intell.*, **11**, pp. 674-693, July 1989.

3. J.M. Combes, et. al., eds., *Wavelets, Time-Frequency Methods and Phase Space*. Springer-Verlag, 1989.

4. A. N. Akansu and R. A. Haddad, *Multiresolution Signal Decomposition: Transforms, Subbands and Wavelets*. Academic Press Inc., 1992.

5. C. K. Chui, *An Introduction to Wavelets*. Academic Press Inc., 1992.

6. C. K. Chui, *Wavelets: A Tutorial in Theory and Applications. Academic Press Inc., 1992.*

7. P. P. Vaidyanathan, *Multirate Systems and Filterbanks*. Prentice-Hall, 1993.

8. A. Papoulis, *Signal Analysis*. McGraw-Hill, 1977.

9. M. J. T. Smith, and T. P. Barnwell, "A Procedure for Designing Exact Reconstruction Filter Banks for Tree-Structured Sub-band Coders," *Proc. IEEE ICASSP*, pp. 27.1.1-27.1.4, 1984.

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