A Generalized Parametric PR-QMF Design Technique Based on Bernstein Polynomial Approximation

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Abstract—**A generalized parametric PR-QMF design technique based on Bernstein polynomial approximation in the magnitude square domain is developed in this paper. The parametric nature of this solution provides useful insights to the PR-QMF problem. Several well-known orthonormal wavelet filters, PR-QMF's, are shown to be the special cases of the proposed technique. Energy compaction performance of a few popular signal decomposition techniques are presented for AR(1) signal sources. It is observed that the hierarchical QMF filter banks considered here outperform the block transforms as expected.**

I. Introduction

Subband signal decomposition techniques have emerged recently as an alternative to the well-known block transforms [1]–[6]. Since the number of bands, M, is equal to the duration, L, of transform basis functions in block transforms, there is not much flexibility to obtain a good frequency split.

Subband filter banks assume $L > M$, therefore a good frequency split is achievable. Obviously, the price paid for this better frequency split is the increase of computational complexity and longer duration in time. Whenever $L = 2M$, the special transform lapped orthogonal transform (LOT) is obtained [7], [8]. In general, there is no constraint on L except the practical considerations.

Perfect reconstruction quadrature mirror filters (PR-QMF) have been proposed as the solution to the two-band frequency split [9]–[11]. These filters, employing a hierarchical tree structure, provide a basis for a multiresolution signal representation. Recently, the wavelet transforms have been proposed as a new approach for multiresolution signal decomposition [12]–[15]. It has also been shown that the wavelet and subband signal representation techniques are very closely interrelated [13], [16].

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We introduce in this paper a generalized, parametric PR-QMF design technique based on Bernstein polynomial approximation [17]. This approach tries to approximate given set of sample points of a desired magnitude square function by using Bernstein polynomials. This approximation is mapped onto Z domain as $R(z)$. The corresponding filter function $H(z)$ is obtained from $R(z)$ via factorization. Section II reviews the PR-QMF banks. The maximally flat magnitude square function is given in Section III and related to the well-known orthonormal wavelet filters [13], [16]. Section IV introduces the proposed generalized, parametric PR-QMF design technique based on the Bernstein polynomial approximation. The energy compaction performance of several different signal decomposition techniques for AR(1) signal sources are presented in Section V. The following section discusses the new directions for future research and concludes the paper.

II. Two Channel PR-QMF Bank

The perfect reconstruction requirements of an orthonormal two-band QMF reduces to [10], [16]

$$Q(z) = H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2$$
$$= R(z) + R(-z) \tag{1}$$

where $H(z)$ is a low-pass filter of length $2N$.

Note that $R(z)$ is a spectral density function and hence is represented by a finite series of the form

$$R(z) = \rho(2N-1)z^{2N-1} + \rho(2N-2)z^{2N-2} + \cdots + \rho(0)z^{0} + \cdots + \rho(2N-1)z^{-(2N-1)}. \tag{2}$$

Then

$$R(-z) = -\rho(2N-1)z^{2N-1} + \rho(2N-2)z^{2N-2} - \cdots + \rho(0)z^{0} - \rho(1)z^{-1} + \cdots - \rho(2N-1)z^{-(2N-1)}. \tag{3}$$

Therefore $Q(z)$ consists only of even powers of z. To force $Q(z) = 2$ it suffices to make all even indexed coefficients in $R(z)$ equal to zero, except for $n = 0$.

However, the $\rho(n)$ coefficients in $R(z)$ are simply the samples of the autocorrelation sequence of $h(n)$

$$\rho(n) = \sum_{k=0}^{2N-1} h(k)h(k+n) = h(n) * h(-n) = \rho(-n) \tag{4}$$

where $*$ indicates a convolution operation.

Now, we need to set $\rho(n) = 0$ for n even, and $n \neq 0$. Therefore,

$$\rho(2n) = \sum_{k=0}^{2N-1} h(k)h(k+2n) = \delta_n. \tag{5}$$

If $x(n)$ is a real sequence and input to a linear, shift-invariant system, the output of the system is expressed as

$$y(n) = x(n) * h(n) \tag{6}$$

where $h(n)$ is the unit sample response of that system. The power spectral density function of the output can be written as

$$P_{yy}(\omega) = |H(e^{j\omega})|^2 P_{xx}(\omega) \tag{7}$$

and the variance of the output signal is obtained as

$$\sigma_y^2 = R_{yy}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 P_{xx}(\omega)\, d\omega. \tag{8}$$

It is seen from (8) that the output energy of the system is a function of input spectral density function and the magnitude square function. This relation is utilized to calculate the band variances which are needed for the energy compaction studies given in Section V.

III. Maximally Flat Magnitude Square Response

Let us assume that $h(n)$ is a length $2N$ low-pass filter with the system function

$$H(z) = \sum_{n=0}^{2N-1} h(n)z^{-n} \tag{9}$$

and its magnitude square function with normalized sampling period

$$|H(e^{j\omega})|^2 = H(z)H(z^{-1}) = \rho(0) + 2\sum_{n=1}^{2N-1} \rho(n)\cos(n\omega). \tag{10}$$

Let the sequence $\rho(n)$ satisfy the following conditions in frequency domain [19]:

$$|H(e^{j\omega})|^2_{\omega=0} = 1 \tag{11}$$

$$\frac{d^\nu}{d\omega^\nu}|H(e^{j\omega})|^2_{\omega=0} = 0 \qquad \nu = 1, 2, \cdots, 2(2N-1-k)+1 \tag{12}$$

$$\frac{d^\mu}{d\omega^\mu}|H(e^{j\omega})|^2_{\omega=\pi} = 0 \qquad \mu = 0, 1, \cdots, 2k-1 \tag{13}$$

where k is an integer to be chosen arbitrarily within the limits $1 \leq k \leq 2N-1$. The parameter k defines the degrees of flatness in the magnitude square function at $\omega = 0$ and at $\omega = \pm\pi$.

If one defines a mapping

$$\cos\omega = 1 - 2x$$

$|H(e^{j\omega})|^2$ can be obtained as a polynomial of degree $2N - 1$

$$P_{2N-1,k}(x) = \sum_{\nu=0}^{2N-1} a_\nu x^\nu \tag{14}$$

with an approximation interval $0 \leq x \leq 1$ and the properties:

a) $P_{2N-1,k}(x)$ has zeros of order k at $x = 1$,
b) $P_{2N-1,k}(x) - 1$ has zeros of order $2N - k$ at $x = 0$.

$P_{2N-1,k}(x)$ in (14) with the conditions a) and b) is a special case of Hermite interpolation problem and it can be solved by using the Newton interpolation formula [17]. But there exists an explicit solution of this problem as given by the expression [19]

$$P_{2N-1,k}(x) = (1-x)^k \frac{1}{(1-k)!}\frac{d^{k-1}}{dx^{k-1}} \sum_{\nu=0}^{2N-2} x^\nu = (1-x)^k \sum_{\nu=0}^{2N-k-1} \binom{k+\nu-1}{\nu} x^\nu. \tag{15}$$

The relation between the autocorrelation sequence of $h(n)$ and the polynomial coefficients a_ν is given by

$$\rho(0) = \frac{1}{2}\sum_{k=0}^{\lceil 2N-1/2 \rceil}\left[2^{-2k}\binom{2k}{k}\sum_{i=2k}^{2N-1} 2^{-i}\binom{i}{2k} a_i\right] \tag{16}$$

and

$$\rho(l) = \sum_{k=0}^{\lceil 2N-1-l/2 \rceil}\left[2^{-(2k+l)}\binom{2k+l}{k} \cdot \sum_{i=2k+l}^{2N-1} 2^{-i}\binom{i}{2k+l} a_i\right] \qquad l = 1, 2, \cdots, 2N-1 \tag{17}$$

where $\lceil x \rceil$ means the integer part of x.

It is clear that this relation provides a simple filter design tool based on the desired degrees of flatness of magnitude square function at $\omega = 0$ and $\omega = \pm\pi$.

It has been stated that if one desires to design a two-band quadrature mirror filter (QMF) bank, the perfect reconstruction of the signal after the synthesis stage requires that

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2.$$

By inspection, the unique maximally flat magnitude square function of PR-QMF has an equal number of zeros at $\omega = 0$ and at $\omega = \pm\pi$. This implies the symmetry of

$|H(e^{j\omega})|^2$ around $\omega = \pi/2$ and expressed as

$$\left.\frac{d^{\mu}|H(e^{j\omega})|^2}{d\omega^{\mu}}\right|_{\omega=0} = 0$$
$$\left.\frac{d^{\mu}|H(e^{j\omega})|^2}{d\omega^{\mu}}\right|_{\omega=\pm\pi} = 0 \qquad \mu = 1, 2, \cdots, 2N-1. \tag{18}$$

Therefore, from (15), $P_{2N-1,k}(x)$ for this special case becomes [19], [13]

$$P_{2N-1,k}(x) = (1-x)^N \sum_{\nu=0}^{N-1} \binom{N+\nu-1}{\nu} x^{\nu}. \tag{19}$$

Using the inverse mappings $x = 1/2\,(1 - \cos\omega)$ and $\cos\omega = 1/2\,(z + z^{-1})$ in (19) the maximally flat magnitude square function can be expressed as [20], [16]

$$H(z)H(z^{-1}) = z^{2N-1}\frac{(1+z^{-1})^{2N}}{4^{2N-1}}\sum_{l=0}^{N-1}\binom{2N-1}{l} \cdot (1+z^{-1})^{2(N-1-l)}(-1)^l(1-z^{-1})^{2l}. \tag{20}$$

It is interesting that the right-hand side of this equation can also be expressed as the linear combination of even indexed, even symmetrical, N binomial sequences of length $(4N-1)$. Therefore, the time domain version of (20) is written easily as

$$\rho(n) = h(n) * h(-n) = \sum_{i=0}^{N-1}\frac{(-1)^i}{4^{2N-1}}\binom{2N-1}{i} x_{2i}(n) \qquad n = 0, 1, \cdots, 4N-2 \tag{21}$$

where [16]

$$x_r(k) = \binom{4N-2}{k}\sum_{\nu=0}^{r}(-2)^{\nu}\binom{r}{\nu}\frac{k^{(\nu)}}{(4N-2)^{(\nu)}} \qquad r, k = 0, 1, \cdots, 4N-2. \tag{22}$$

IV. A Generalized PR-QMF Design Technique Using Bernstein Polynomial Approximation

Two-band orthonormal PR-QMF requires that the magnitude square condition, (1),

$$|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2 = 2$$

be satisfied where $|H(e^{j\omega})|^2$ is the magnitude square function of the low-pass filter to be designed with the length $2N$. Since $\rho(2n) = \delta_n$ holds for an orthonormal or paraunitary PR-QMF, (10) is easily modified for this case as

$$|H(e^{j\omega})|^2 = 1 + 2\sum_{k=1}^{N}\rho(2k-1)\cos(2k-1)\omega \tag{23}$$

with the nonnegativity constraint

$$|H(e^{j\omega})|^2 \geq 0, \qquad \forall\omega.$$

Let $f(x)$ be defined on the interval [0, 1]. The Nth ($N \geq 1$) order Bernstein polynomial approximation to $f(x)$ is expressed as [17]

$$B_N(f; x) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} x^k(1-x)^{N-k}. \tag{24}$$

Equation (24) indicates that the interval [0, 1] is divided into N equal subintervals. Only the samples of $f(x)$ at those $(N+1)$ points are used to obtain the approximation $B_N(f; x)$. If $f(x)$ is differentiable, the approximation is also valid for its differentials. This implies that

$$B_N(f; x) \to f(x)$$
$$B'_N(f; x) \to f'(x)$$

where prime means the derivative. This feature holds true also for higher derivatives. Therefore the Bernstein polynomials provide simultaneous approximations of a function and its derivatives [17].

It is interesting that a monotonic and convex function is approximated by a monotonic and convex approximant if Bernstein polynomials are used. This tells us that the approximation follows the behavior of the approximated function to a remarkable degree. The price paid for this nice feature is that these polynomials converge slowly to the function to be approximated.

The proposed two-band PR-QMF design approach is introduced now. This approach provides the tools to design PR-QMF's based on the desired magnitude square functions in continuous variable x. The samples of the desired function are approximated by the Bernstein polynomials.

Let us consider now a desired low pass function $f(x)$, $0 \leq x \leq 1$, which satisfies the PR-QMF magnitude square conditions in x

$$f(x) + f(1-x) = 1, \qquad f(x) \geq 0. \tag{25}$$

As in the ideal low-pass case if we choose

$$f(x) = \begin{cases} 1, & 1 \leq x \leq 1/2 \\ 0, & 1/2 < x \leq 1 \end{cases} \tag{26}$$

its equidistant sample values at $2N$ sampling points are stated as

$$f\left(\frac{i}{2N-1}\right) = \begin{cases} 1 & 0 \leq i \leq N-1 \\ 0 & N \leq i \leq 2N-1 \end{cases}. \tag{27}$$

For interpolating these samples of the desired function $f(x)$ with the Bernstein polynomials, substitute (27) into (24) and the corresponding interpolation function is found as

$$B_{2N-1}(f; x) = \sum_{i=0}^{N-1}\binom{2N-1}{i} x^i(1-x)^{2N-1-i} = (1-x)^N\sum_{i=0}^{N-1}\binom{N+i-1}{i} x_i. \tag{28}$$

The properties of the Bernstein polynomials assure that

$$0 \leq \min\left\{f\left(\frac{i}{2N-1}\right)\right\} \leq B_{2N-1}(f; x) \leq \max\left\{f\left(\frac{i}{2N-1}\right)\right\} \leq 1$$

holds for the interpolation considered here [17]. It is seen from (28) that the interpolation function $B_{2N-1}(f; x)$ is the maximally flat function with the symmetry around $x = 1/2$ within the interval $0 \leq x \leq 1$. This is the unique maximally flat magnitude square function of PR-QMF's in the variable x, as was shown in (20).

If one maps x onto Ω, $0 \leq \Omega \leq \infty$ as [18]

$$x = \frac{\Omega^2}{1+\Omega^2}$$

then the corresponding rational function in Ω is found as

$$B_{2N-1}(f; \Omega) = \frac{\sum_{i=0}^{N-1} \binom{2N-1}{i} \Omega^{2i}}{(1+\Omega^2)^{2N-1}}. \quad (29)$$

If we now define $q = j\Omega$ and use the conformal mapping

$$z = \frac{1+q}{1-q}$$

then the magnitude square function mapped into the z domain is obtained as

$$\begin{aligned} R(z) &= z^{2N-1} \frac{(1+z^{-1})^{2N}}{4^{2N-1}} \sum_{i=0}^{N-1} (-1)^i \binom{2N-1}{i} \\ &\cdot (1+z^{-1})^{2(N-1-i)}(1-z^{-1})^{2i} \\ &= H(z)H(z^{-1}). \end{aligned} \quad (30)$$

The magnitude square function $R(z)$ is factorized to obtain the PR-QMF low-pass filter $H(z)$. This low-pass filter $H(z)$ is identical to the binomial-QMF. It is shown in [16] that binomial-QMF is also identical to the compactly supported orthonormal wavelet filters proposed by Daubechies [13]. Therefore, here we connect the works of Herrmann [19] and Daubechies [13] as the special case of the proposed PR-QMF design technique.

Remark 1: $R(z)$ corresponds to a low-pass function with $R(e^{j\pi/2}) = R(e^{j0})/2$. It is expressed as a combination of odd harmonics of the cosine functions. These coefficients of the representation also correspond to the Fourier coefficients of the ideal low-pass function.

The PR-QMF filters obtained from an orthonormal wavelet approach have the regularity or differentiability constraint of wavelet functions in addition to the PR-QMF constraints. Daubechies [13] has proposed to use the zeros of function $H(z)$ at $\omega = \pi$ as a tool to have some regularity. The filters with maximally flat magnitude square function have maximum possible zeros at $\omega = \pi$ but they are not the most regular solutions [21].

Now, we extend the proposed design technique to obtain a broad family of smooth PR-QMF's defined by a set of parameters. If one defines a set of nonincreasing, positive function samples which will be the guide points of the approximation as

$$f\left(\frac{i}{2N-1}\right) = \begin{cases} 1 & i = 0 \\ 1-\alpha_i & 1 \leq i \leq N-1 \\ \alpha_i & N \leq i \leq 2(N-1) \\ 0 & i = 2N-1 \end{cases} \quad (31)$$

where $\alpha_i = \alpha_{2N-1-i}$ and $0 \leq \alpha_i < 0.5$ with $1 \leq i \leq N-1$ then the approximation to $f(x)$ with those constraints of (25) using the Bernstein polynomials is expressed as

$$\begin{aligned} B_{2N-1}(f; x) = &\sum_{i=0}^{N-1} \binom{2N-1}{i} x^i (1-x)^{2N-1-i} \\ &- \sum_{i=1}^{N-1} \alpha_i \binom{2N-1}{i} x^i (1-x)^{2N-1-i} \\ &+ \sum_{i=N}^{2(N-1)} \alpha_i \binom{2N-1}{i} x^i (1-x)^{2N-1-i}. \end{aligned} \quad (32)$$

After applying similar mappings, the corresponding magnitude square function in z domain, $R(z)$, is obtained as

$$\begin{aligned} R(z) = z^{2N-1} \frac{(1+z^{-1})^{2N}}{4^{2N-1}} \Bigg\{ &\sum_{i=0}^{N-1} (-1)^i \binom{2N-1}{i} \\ &\cdot (1+z^{-1})^{2(N-1-i)}(1-z^{-1})^{2i} \\ &- \sum_{i=1}^{N-1} (-1)^i \alpha_i \binom{2N-1}{i} \\ &\cdot (1+z^{-1})^{2(N-1-i)}(1-z^{-1})^{2i} \\ &+ \sum_{i=N}^{2(N-1)} (-1)^i \alpha_i \binom{2N-1}{i} \\ &\cdot (1+z^{-1})^{2(N-1-i)}(1-z^{-1})^{2i} \Bigg\}. \end{aligned} \quad (33)$$

Similarly, the low-pass PR-QMF function $H(z)$ is obtained from the magnitude square function $R(z)$ via factorization.

Example: We design a 6-tap smooth PR-QMF with the desired sample values in magnitude square domain

$$f\left(\frac{i}{2N-1}\right) = \begin{cases} 1 & 0 \leq i \leq 1 \\ 1-\alpha & i = 2 \\ \alpha & i = 3 \\ 0 & 4 \leq i \leq 5 \end{cases} \quad (34)$$

where $0 \leq \alpha < 0.5$. This set of desired samples in frequency corresponds to a low-pass filter function $h(n)$. Its high-pass mirror filter has two vanishing moments for $\alpha > 0$, and three vanishing moments for $\alpha = 0$ [13], [16]. The magnitude square function for this example is obtained similarly as

$$R(z) = z^5 \frac{(1+z^{-1})^6}{4^5} \left\{ \sum_{i=0}^{2} (-1)^i \binom{5}{i} \cdot (1+z^{-1})^{2(2-i)}(1-z^{-1})^{2i} - \alpha \binom{5}{2} (1-z^{-1})^4 - \alpha \binom{5}{3} \cdot (1+z^{-1})^{-2}(1-z^{-1})^6 \right\}. \tag{35}$$

From there, one can obtain the 6-tap low-pass PR-QMF $H(z)$ via factorization. Fig. 1 displays $f(i/2N - 1)$, $B_{2N-1}(f; x)$ and $R(z)$ functions for the 6-tap PR-QMF case with $\alpha = 0.25$.

Remark 2: The vanishing moments of PR-QMF high-pass filter $(-1)^{n+1}h(1-n)$ with length $2N$ is defined in the time domain as [21]

$$\sum_n n^i (-1)^{n+1} h(1-n) = 0 \qquad i = 0, 1, \cdots, N.$$

This is equivalent to the flatness requirement of the QMF filter response in frequency

$$\left.\frac{d^\nu H(e^{j(\omega+\pi)})}{d\omega^\nu}\right|_{\omega=0} = 0 \qquad \nu = 0, 1, \cdots, N.$$

It is seen that the maximally flat magnitude square filters, binomial-QMF, with $\nu = N$, has the maximum number of vanishing moments on their high-pass filter. It is clear that all possible moments are used only by the high-pass filter and the low-pass filter does not have any vanishing moments. $\alpha_i \neq 0$ for any i decreases the number of vanishing moments of the high-pass filter by one in the proposed design technique.

The magnitude functions of several known smooth or regular 6-tap QMF's and their α values, as defined in the earlier Example, are displayed in Fig. 2.

Remark 3: It is found that $\alpha = 0.2708672$ corresponds to the 6-tap Coiflet filter solution and $\alpha = 0.0348642$ gives the 6-tap PR-QMF of the most regular orthonormal wavelet solution [21]. $\alpha = 0$ gives the binomial QMF-wavelet transform with three vanishing moments [13], [16]. This parametric solution of the PR-QMF problem provides a useful tool for the design of orthonormal wavelet bases. It connects the frequency behavior of the PR-QMF with the properties of the corresponding wavelet transform basis. Since this subject is beyond the focus of the paper, it will be reported in another paper.

V. Energy Compaction and Performance Results

All orthonormal, variance preserving, signal decomposition techniques can be evaluated by employing the energy compaction criterion. For the block transforms, gain of transform coding over PCM, G_{TC}, and for the subband decomposition techniques, gain of subband coding over PCM, G_{SBC} are unified as [22]

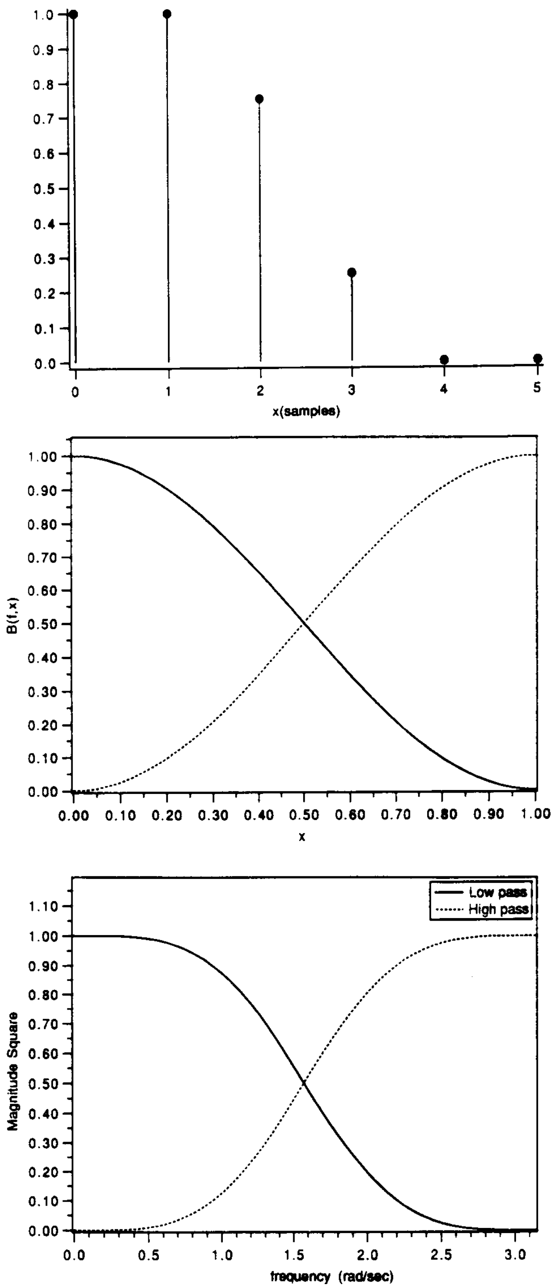

Fig. 1. $f(i/2N - 1)$, $B_{2N-1}(f; x)$, and $R(z)$ functions for the 6-tap PR-QMF case with $\alpha = 0.25$.

$$G = \frac{\frac{1}{M}\sum_{l=1}^{M} \sigma_l^2}{\left[\prod_{l=1}^{M} \sigma_l^2\right]^{1/M}}. \tag{36}$$

Here σ_l^2 is the variance of the lth transform coefficient or the lth subband of the analysis stage. This equation holds

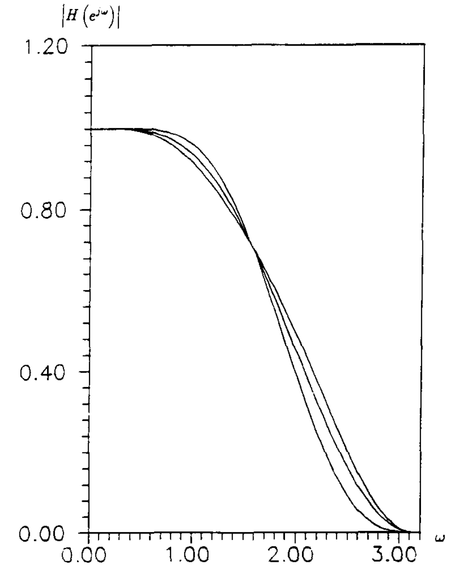

Fig. 2. Magnitude functions of three different 6-tap PR-QMF's; maxflat (α = 0), coiflet of [21] (α = 0.2708672), and for α = 0.480.

TABLE I
ENERGY COMPACTION OF DCT, KLT, 6 AND 8-TAP BINOMIAL-QMF [16], [13], 8-TAP SMITH-BARNWELL CQF [9], AND IDEAL FILTER BANK AS A FUNCTION OF M FOR AR (0.95)

M	DCT	KLT	6-Tap B-QMF	8-Tap B-QMF	8-Tap Smith-Barnwell	Ideal
2	3.20	3.20	3.76	3.81	3.83	3.94
4	5.71	5.73	6.77	6.90	6.97	7.23
8	7.63	7.66	8.52	8.74	8.84	9.16
16	8.82	8.86	9.25	9.50	9.62	9.95
...	...	...	...	...	...	...
∞	10.25	10.25	10.25	10.25	10.25	10.25

for regular binary tree structures of subband technique. The derivation of G assumes the same pdf type for all coefficients or bands as well as the input. This criterion is derived by using the rate-distortion theory. It is already widely employed in the signal processing and coding fields as a common performance tool for different signal decomposition techniques [23].

The unidimensional first-order autoregressive signal model AR (1) is used for energy compaction comparisons of several decomposition techniques. This source model is defined as

$$s(n) = rs(n-1) + \xi(n) \tag{37}$$

where r is the correlation coefficient and $\xi(n)$ is a zero-mean white Gaussian noise with known variance. This model is a crude approximation to the real world signals like speech and images, $r = 0.85$ for speech, $r = 0.95$ for image, and commonly used for performance comparisons. Energy compactions G, of several decomposition techniques such as DCT, KLT, binomial-QMF [16], and Smith–Barnwell CQF [9] based hierarchical subband structures and ideal subband filter banks for different values of M and $r = 0.95$ are given in Table I. It is seen from this table that the ideal filter bank reaches the performance upper bound, when $M \to \infty$, faster than the optimum block transform KLT. It is also observed that the performance of 8-tap binomial-QMF and Smith–Barnwell CQF are very comparable for the cases considered here.

Fig. 3 displays the 2-band energy compaction performance of 4-tap and 6-tap 2-band PR-QMF banks as a function of α for the input sources AR (0.75), AR (0.85), and AR (0.95). It is seen that $\alpha = 0$, the binomial QMF, compacts better than all the other smooth QMF solutions. It is expected for this source model since $\alpha = 0$ corresponds to the maximally flat magnitude square function around $\omega = 0$. Fig. 4 displays the variations of the filter coefficients of all the possible smooth PR-QMF's as a function of α for $2N = 4$. Haar basis corresponds to $\alpha = 1/3$ in this figure. The phase responses of the filters in this α range are linear-like. Similarly, Fig. 5 gives the coefficients of all possible 6-tap smooth PR-QMF's as a function of parameter α. Here the high-pass filters have two vanishing moments except for $\alpha = 0$.

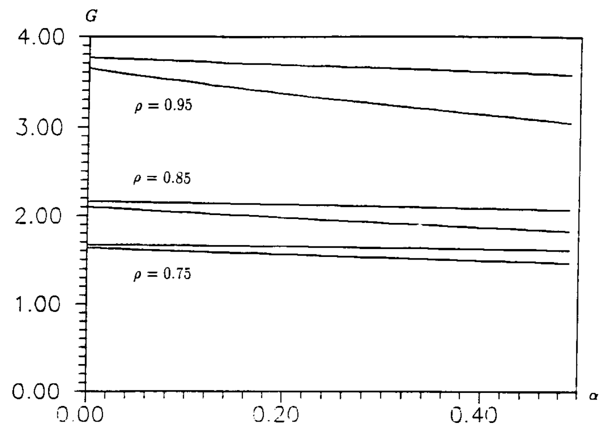

Fig. 3. Two-band energy compaction of 4 and 6-tap PR-QMF's as a function of α for AR (0.75), AR (0.85), and AR (0.95) sources.

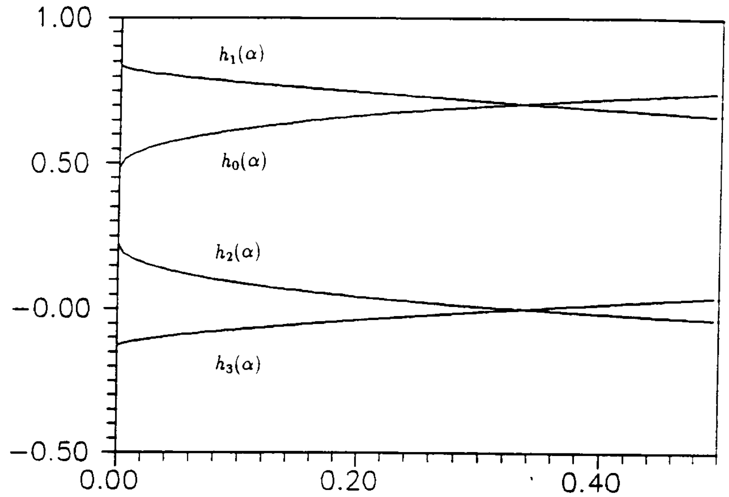

Fig. 4. All the possible smooth 4-tap PR-QMF coefficients as a function of α.

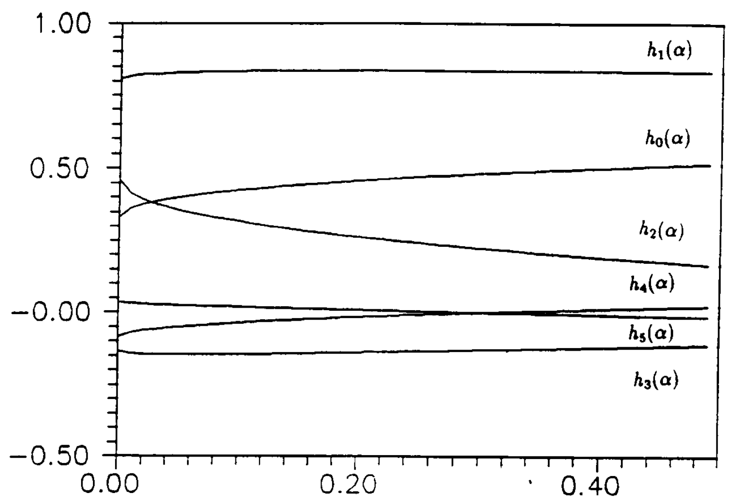

Fig. 5. All the possible smooth 6-tap PR-QMF coefficients as a function of α.

VI. Discussion and Conclusions

A parametric PR-QMF design technique based on the Bernstein polynomial approximation is developed in this paper. Any orthonormal PR-QMF can be designed with this technique. The filter examples we considered are the special cases with smooth, ripple-free frequency responses. This design approach can also be used for rippled QMF design problems. All PR-QMF filter solutions of a given length can be obtained as a function of the design parameters in this approach.

Since the wavelet functions are evaluated with their regularity, this approach provides a tool to design orthonormal wavelet bases with the desired degree of regularity [13], [21]. For this purpose, the parameters of the design technique are linked to the regularity of the corresponding wavelet function. This provides the pattern of the relations between the frequency behavior of PR-QMF's and the degree of regularity or differentiability of the corresponding orthonormal wavelet functions.

From the signal coding point of view, the energy compaction is an important performance measure. It is shown that the max-flat magnitude square binomial QMF-wavelet filters [13], [16] and Smith-Barnwell CQF [9] have comparable energy compaction and both outperform the most regular wavelet filters [21] for the cases considered here.

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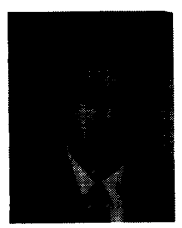

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