ON BASIC PRICE MODEL AND VOLATILITY IN MULTIPLE FREQUENCIES

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ABSTRACT

This paper revisits volatility and emphasizes interrelationships of risk metrics at various time horizons expressed in multiple frequencies. The basic price model defined by Black-Scholes equation and its extensions for varying variance scenarios are presented, i.e. Heston and GARCH models. Moreover, we highlight the significance of abrupt changes in the price of an asset on price modeling and volatility estimation. We extend basic price model where price jumps are taken into account as well. The proposed approach is validated by simulations, and shown that it improves volatility estimation.

Index Terms— Price Models, Black-Scholes, Volatility Models, Price Jumps and Regime Change, Multiple Frequency Finance

1. INTRODUCTION

The basic price model has been widely used in classical econometrics and finance as a framework to describe the relationship between return and various measures of risk for a financial instrument. Volatility quantifies the risk of an investment, and commonly related to the standard deviation of continuously compounded returns of an asset for a defined time interval, i.e. horizon of investment. Daily, monthly, and annual volatilities are commonly used in finance.

If the price of an asset changes according to a Gaussian random walk, the distribution of the process becomes more spread as time increases. In contrast, the increase of volatility for the same equity is related to the square root of time horizon. Several researchers also studied Levy distribution as an alternative to Gaussian since it represents "fat tails" with finite variance better than the latter. Note that Black-Scholes equation [1] assumes a predictable constant volatility that is not the case in most times. The local volatility [2] and stochastic volatility [3, 4] are quite popular among many in the literature to model volatility.

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2. VOLATILITY OF AN ASSET

The continuously compounded return (logarithmic return) of an asset is defined as

$$r = \ln\left(S_f/S_i\right)$$

where S_i and S_f are its initial and final values, respectively. The volatility of the asset is the standard deviation of its return defined as

$$\sigma = \left(E\left[r^2\right] - E^2\left[r\right]\right)^{1/2}$$

The annualized volatility, σ_a, is the standard deviation of the yearly compounded returns of an investment. The generalized volatility for the time interval of T in years is expressed as $\sigma_T = \sigma_a\sqrt{T}$. Similarly, the annualized volatility for the standard deviation of the daily compounded returns, σ_D, and the time period of D days is written in $\sigma_a = \sigma_D\sqrt{D}$ where D is the number of business days in a year, and it is typically 252 in USA. For example, the annual volatility for a daily volatility of $\sigma_D = 10^{-4}$ (1 basis point, also known as 1 bps) is calculated accordingly, $\sigma = 10^{-4}\sqrt{252} \cong 0.16\%$. Hence, one can calculate volatilities for various time resolutions as follows

$$\begin{aligned} \sigma_{month} &= \sigma_a\sqrt{1/12} \\ \sigma_{hour} &= \sigma_a\sqrt{1/\left(252\times 6.5\right)} \end{aligned} \quad (1)$$

where we assume the market is open for 6.5 hours a day which is the case for the US markets. The multiple frequency measures of volatility given in Eq. 1 are valid for the assumption of Gaussian random walk in price formation. In general, those relationships are more involved mathematically for stochastic modeling of real world signals. Some researchers proposed Levy stability exponent α for better modeling of multiresolution volatilities expressed as $\sigma_T = \sigma T^{\frac{1}{\alpha}}$ where $\alpha = 2$ yields Gaussian random walk.

3. BASIC PRICE MODEL

The price of an equity with constant volatility assumption satisfies the stochastic differential equation of the geometric Brownian motion written as

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \quad (2)$$

where $S(t)$ is the price of the asset, $dS(t)$ is the differential of $S(t)$, μ is normalized (percentage) constant drift, σ is normalized constant volatility, and $W(t)$ is a Wiener process or Brownian motion such that $dW(t)$ is a normal process. This stochastic differential equation has its analytic solution obtained by using Ito's Lemma [5, 6] and expressed as

$$S(t) = S(0)e^{(\mu - 1/2\sigma^2)t + \sigma W(t)} = S(0)e^{R(t)} \tag{3}$$

where $R(t)$ is the return of an investment at time t from its original price of $S(0)$. Note that the expected value and variance of $S(t)$ are expressed as $E[S(t)] = S(0)e^{\mu t}$ and $\text{var}[S(t)] = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1)$, respectively. The normalized price at time t is defined as

$$\overline{S}(t) \triangleq S(t)/S(0) = e^{(\mu - 1/2\sigma^2)t + \sigma W(t)}$$

where $S(0)$ is the arbitrary initial price. Note that the normalized price $\overline{S}(t)$ is a log-normally distributed random variable where $N = \ln \overline{S}(t)$ is a normal process with $E[N] = \mu t$ and $\sigma_N^2 = \sigma^2 t$. The basic price model with constant σ holds for non-stochastic volatility models like Black-Scholes. For the case of stochastic volatility models we replace σ^2 with a volatility function $v(t)$ as the model of varying $\sigma^2_{S(t)}$. $v(t)$ is also commonly modeled as Brownian motion and its form depends on the stochastic volatility model employed as follows

$$dv(t) = \alpha_{S(t)}dt + \beta_{S(t)}dB(t)$$

where $\alpha_{S(t)}$ and $\beta_{S(t)}$ are functions of $v(t)$. $dB(t)$ is also normal process and it is correlated to $dW(t)$ with correlation coefficient ρ. Therefore, the basic price model of Eq. 2 can be rewritten as

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dW(t) \tag{4}$$

The two popular stochastic volatility models are Heston and Generalized Auto-regressive Conditional Heteroskedasticity (GARCH) models. According to Heston stochastic volatility model [2], the randomness of the variance process changes with the square root of variance in the model. Hence, the stochastic differential equation of Eq. 4 for variance is expressed as follows

$$dv(t) = \theta\left[w - v(t)\right]dt + \xi\sqrt{v(t)}dB(t)$$

where w is the mean long term volatility, θ is the rate at which the volatility reverts itself toward long term mean, ξ is the volatility of the volatility process, and $dB(t)$ is a normal process, and $dW(t)$ and $dB(t)$ are correlated with correlation coefficient ρ. Therefore, Heston stochastic volatility model assumes that a) volatility exhibits a tendency to revert towards w at rate θ, b) volatility has a constant volatility ξ of its own, c) the randomness of volatility is correlated to the randomness of underlying price process with correlation coefficient ρ.

Generalized Auto-regressive Conditional Heteroskedasticity (GARCH) [3, 4] is another popular model to estimate stochastic volatility of an asset. It assumes that the randomness of the variance process changes with the variance as expressed in standard GARCH(1,1) model

$$dv(t) = \theta\left[w - v(t)\right]dt + \xi v(t)dB(t)$$

It is a well known fact that asset values change and go through various high and low volatility intervals during their life spans. The periods when prices fall abruptly are usually followed by additional drops or by unexpectedly strong price increases. Similarly, a period of price increase may be followed by another period of increase or by a period of significant drop in price. In contrast, in some periods of time prices might move up and down slowly, or does not change much at all. It is commonly agreed in finance that the extreme price changes have their drivers and they don't happen randomly. They are caused by above the average movements in the markets, and creating heteroskedasticity with time varying volatility in asset prices.

4. BASIC PRICE MODEL IN DISCRETE TIME

Although the basic price model described in previous section is defined in continuous time variable t, in reality, the price changes happen at discrete time points. Let's define the sampling time interval as T_s where price information in discrete-time is expressed as

$$s(n) \triangleq S(nT_s) = S(t)|_{t=nT_s}$$

where T_s is typically in the range of 1 microsecond for high-frequency trading scenarios. The price at discrete-time $n+1$ is modeled as the summation of price at n and a Gaussian random variable $w(n) \sim \mathcal{N}(\mu, \sigma)$. Hence, we model price change within the sampling interval T_s as

$$s(n+1) = s(n) + w(n) \tag{5}$$

Similarly, we can express the next price at discrete-time $n+2$ as

$$s(n+2) = s(n+1) + w(n+1) = s(n) + w(n) + w(n+1)$$

Therefore, by using the central limit theorem, one can express the future price at $T = MT_s$ as a discrete-time approximation to Eq. 2 for

$$S(t) \cong s(n+M) = s(n) + \sum_{k=0}^{M-1} w(n+k) \triangleq s(n) + W(n+M) \tag{6}$$

where $W(n+M) \sim \mathcal{N}(\mu M, \sigma\sqrt{M})$. In real world, the variance or volatility of price variations is not constant, and this type of stochastic processes are called heteroskedastic. There are several mathematical models to describe heteroskedastic processes reported in the literature. Auto-regressive conditional heteroskedasticity (ARCH) and its varieties are widely used in finance. We briefly describe it in Sec. 5.2.

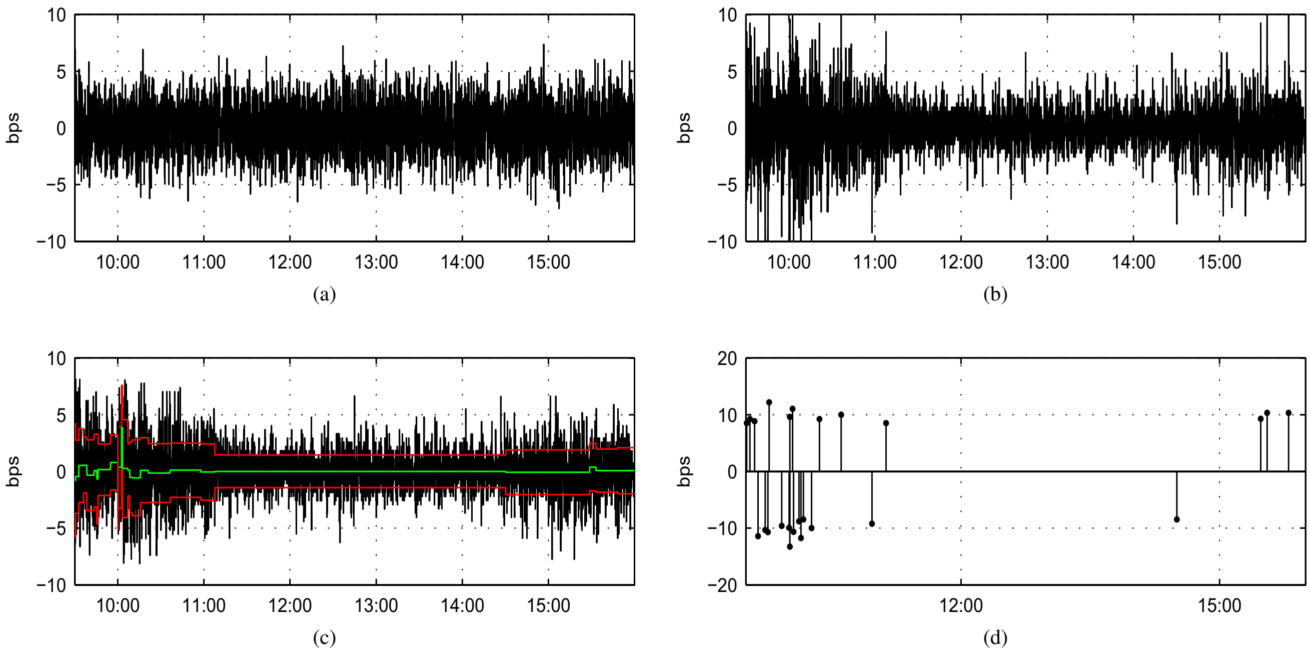

Fig. 1: (a) A realization of a white Gaussian random process with $\mu = 0.01$ bps and $\sigma = 2.11$ bps, (b) Returns of Apple Inc. (AAPL) stock on 6/17/2010. It is clear that the return process is not white Gaussian, (c) artificial jump-free return process generated by removing the jumps from the AAPL returns (black,) its mean (green,) and standard deviation (red) w.r.t. its mean, both measured for each different "regime" defined by the "jumps," (d) Jumps of AAPL returns.

5. BASIC PRICE MODEL WITH JUMPS

The price of an asset is impacted by many reasons including stock specific and stock related business developments and financial news. Although some of those news are anticipated, there are many instances that these higher impact events happen quite randomly. One may observe upward and downward abrupt price changes of any asset. The abrupt changes are called as jumps in finance literature[7]. Therefore, the basic price model with jumps fits real-world price data better and it is expressed as follows

$$s(n) = s(n-1) + j(n) + N(n) \tag{7}$$

where $j(n) \in \mathbb{R}$ is the abrupt price change, up or down, that happens at discrete-time n, and $N(n) \sim \mathcal{N}(\mu, \sigma)$ is a Gaussian random process. Note that in Eq. 7 we model the random return $w(n)$ of Eq. 5 as the summation of two processes. Namely, jump process $j(n)$, and a pure Gaussian noise process $N(n)$,

$$w(n) = j(n) + N(n) \tag{8}$$

In Fig. 1b, returns of Apple Inc. (AAPL) stock on day 6/17/2010 at $T_s = 5s$ are displayed. For this case, estimated mean (drift) and standard deviation (volatility) of the returns are calculated as ~0.01 bps and ~2.11 bps, respectively. In Fig. 1a a realization of a Gaussian random process $\mathcal{N}(\mu, \sigma^2)$ with $\mu = 0.01$ bps and $\sigma = 2.11$ bps is shown. It is observed from Figs. 1a and 1b that one needs to consider the jump process in the price model in order to employ the basic price model more properly. Any jump of high significance is the main reason for the so-called regime change in an asset price.

In Fig. 2a, price of AAPL at $T_s = 5s$ (black) and $T_s = 300s$ (red-dashed) are displayed along with their corresponding means and standard deviations (Eq. 6.) In order to highlight the importance of jump processes in price modeling, we define the volatility estimation error as

$$\epsilon = \left| \hat{\sigma}_m \sqrt{k/m} - \hat{\sigma}_k \right| \tag{9}$$

where $\hat{\sigma}_m$ and $\hat{\sigma}_k$ are the volatilities estimated at $T_s = m$ and $T_s = k$, respectively. If the return process $w(n)$ in Eq. 5 were pure Gaussian, than the error term ϵ would be zero. In the case of Fig. 2a, we calculated $\epsilon = 2.01$ bps. We employed a histogram based price jump detector where a return is labeled as a jump if its absolute value is larger than 4σ. The detected jump process, $j(n)$, along with the jump-free noise process, $N(n)$, are shown in Figs. 1d and 1c. Next, we restart the estimation of drift and volatility at the beginning of each regime. Note that a "regime" starts with the occurrence of a jump and it lasts until another jump is observed. It is observed from Fig. 2b that the proposed approach tracks the price process better

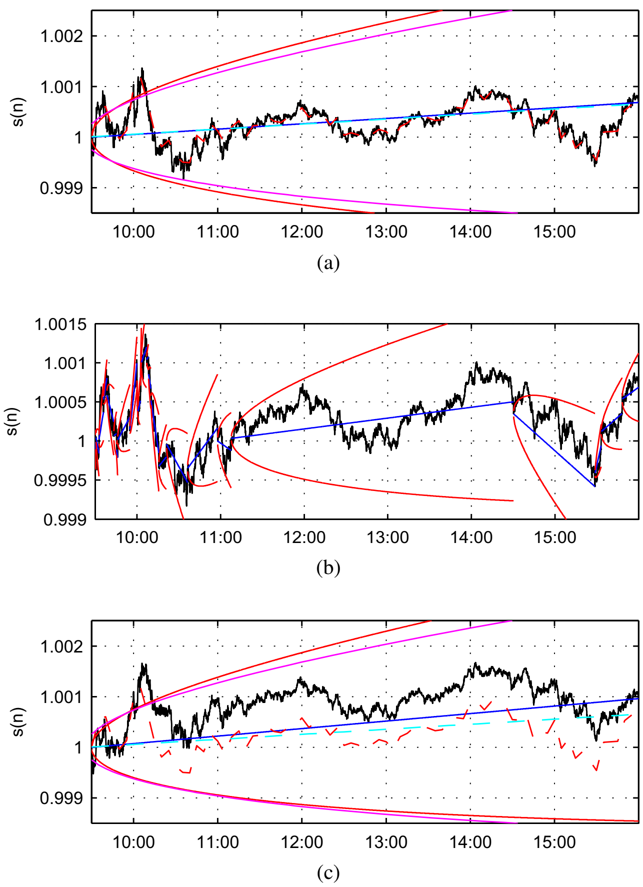

Fig. 2: (a) Price of Apple Inc. (AAPL) stock on 6/17/2010 at $T_s = 5s$ and $T_s = 300s$ (black and red-dashed, respectively) along with their means (blue and cyan-dashed, respectively,) and standard deviations (red and magenta, respectively,) (b) Price at $T_s = 5s$ along with its mean and standard deviation (In this case estimation is restarted at the beginning of each regime), (c) Same as (a) but the price is generated using the jump-free returns defined in Eq. 10. Same color coding as in (a) is used.

as expected. Next, we define an artificial return process as

$$\hat{w}(n) = N(n) = w(n) - j(n) \tag{10}$$

We plot the resulting price process along with its estimated drift and standard deviation in Fig. 2c. In this case, the error is reduced to $\epsilon = 0.98$ bps from $\epsilon = 2.01$ bps, where $\hat{w}(n)$ process is almost jump-free. It is observed from Fig. 2 that jump is an important phenomenon, and one needs to take these abrupt changes into account in order to better model the price process. We repeat this experiment for various frequencies spanning from $k = 1s$ to 300s with $m = 1$ (Eq. 9) for both $w(n)$ and $\hat{w}(n)$. The error ϵ as a function of frequency k is displayed in Fig. 3. It is seen from the figure that removing jumps significantly reduces the volatility estimation error.

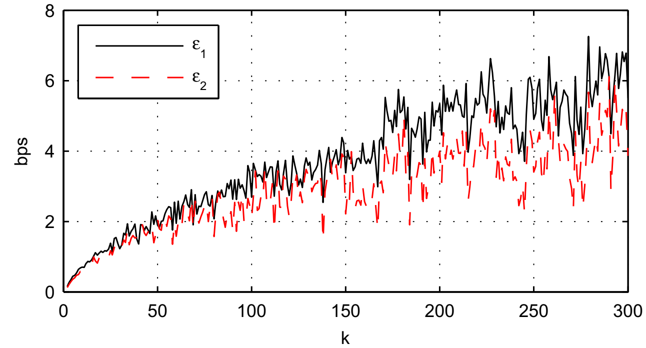

Fig. 3: Volatility estimation error ϵ versus sampling period k with $m = 1$ as defined in Eq. 9 for real and artificial (jump-free) returns of Eqs. 8 and 10 (black and red, respectively).

6. CONCLUSION

In this paper, we revisited basic price model as defined in Black-Scholes equation with constant volatility. After a brief discussion on volatility, and Heston and GARCH volatility models, we highlighted the significance of abrupt changes observed in real-world data. We incorporated these jumps in price model and volatility estimation independently than typical variations. Moreover, we extended the proposed method to multiple frequencies. It was shown that the proposed method offers improved volatility estimation.

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