1 How to find sums without using induction?

Proposition 1 *For all $n \geq 0$,*

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Proof. One can show this proposition by using induction. But what if we don't know how much the sum is? How can we find the answer $n(n+1)/2$?

We use the following trick to find sums of the following form

$$S_k = \sum_{i=0}^{n} i^k.$$

First consider $(i+1)^{k+1}$ and expand it. Substitute in the expansion $i = 1$, $i = 2, \ldots$, $i = n$, a total of n times and write the resulting n equalities one after the other. Then, sum these n equalities by summing up the left hand sides and the right hand sides. Solve for S_k and S_k can then be found as a function of n.

For the sum in question $k = 1$. Therefore we consider

$$(i+1)^2 = i^2 + 2i + 1$$

We substitute for $i = 1, 2, \ldots, n$ writing one equality after the other

$$\begin{aligned}
(1+1)^2 &= 1^2 + 2 \cdot 1 + 1 \\
(2+1)^2 &= 2^2 + 2 \cdot 2 + 1 \\
(3+1)^2 &= 3^2 + 2 \cdot 3 + 1 \\
(4+1)^2 &= 4^2 + 2 \cdot 4 + 1 \\
\ldots &= \ldots \\
(n+1)^2 &= n^2 + 2 \cdot n + 1
\end{aligned}$$

When we sum up the n equalities we realize that say, $(3+1)^2$ of the third line is equal to 4^2 of the fourth line and therefore.

$$(1+1)^2 + (2+1)^2 + \ldots + (n+1)^2 = (1^2 + 2^2 + 3^2 + \ldots + n^2) + 2 \cdot (1 + 2 + \ldots + n) + (1 + \ldots + 1)$$

We note that $2 \cdot (1 + 2 + \ldots + n) = 2S_1$ and $(1 + \ldots + 1) = n$ (number of ones is number of equations). Then,

$$(n+1)^2 = 1 + 2S_1 + n$$

Solving for S_1 we get that $S_1 = ((n+1)^2 - n - 1)/2$, ie $S_1 = (n^2 + 2n + 1 - n - 1)/2 = (n^2 + n)/2 = n(n+1)/2$, which proves the desired result.

Example 1 *For all $n \geq 0$, find*

$$\sum_{i=0}^{i=n} i^2$$

Example 2 *For all $n \geq 0$, find*

$$\sum_{i=0}^{i=n} i^3$$

2 Logarithms and Exponentials

When we describe the performance of computer algorithms we frequently use logarithms base two and powers of two. A brief review of topics related to logarithms and exponentials is given below. For more details, one can review section 3.2 (page 52).

1. **The floor function** $\lfloor x \rfloor$ **:** denotes the largest integer smaller than or equal to x, i.e. $\lfloor 3.5 \rfloor = 3$, $\lfloor -3.5 \rfloor = -4$, and $\lfloor 3.0 \rfloor = 3$.
2. **The ceiling function** $\lceil x \rceil$ **:** denotes the smallest integer greater than or equal to x, i.e. $\lceil 3.5 \rceil = 4$, $\lceil -3.5 \rceil = -3$, and $\lceil 3.0 \rceil = 3$.

Properties of exponentials. Let a, m, n be real numbers such that $a \neq 0$.

3. $a^0 = 1$, $a^1 = a$, $a^{-1} = 1/a$.
4. $a^m \cdot a^n = a^{n+m}$, $a^m / a^n = a^{m-n}$.
5. $(a^m)^n = (a^n)^m = a^{(mn)}$.
6. Let $c \geq 1, d \geq 1$ be constants. There is a constant n_0 such that for all $n \geq n_0$ we have that $c^n > n^d$.

Definitions and Properties of logarithms.

7. The natural (Neperian) logarithm log (also ln) of x denoted by $\log x$ is the real number y such that $e^y = x$. e is the well known constant $e = 2.7172\ldots$. In this course we **prefer to write and use** $\log x$ to $\ln x$.
8. For all real x, $e^x \geq 1 + x$.
9. For all x such that $|x| < 1$, $1 + x \leq e^x \leq 1 + x + x^2$.
10. The base-2 logarithm of x, denoted by $\lg x$ (or sometimes $\log_2 x$), is the real number y such that $2^y = x$.

Properties of base-2 logarithms (See page 53 for generalization to any base other than two).

11. $\lg^k n = (\lg n)^k$. Note that $\lg^{(k)} n$ with a parenthesized exponent means something else (see page 55 of CLRS).
12. $\lg \lg n = \lg(\lg n)$.
13. For all $a > 0, b > 0, c > 0$ and n we have that
    a. $a = 2^{\lg a}$,
    b. $\lg(ab) = \lg a + \lg b$,
    c. $\lg(a/b) = \lg a - \lg b$,
    d. $\lg a^n = n \ \lg a$,
    e. $\lg a = \frac{\log a}{\log 2}$.

Fact 1 *The expression "for large enough n " means "there is a positive constant n_0 such that for all $n > n_0$ ".*

Fact 2 *For any positive constant k, m and integer $n > 0$, we have that $n^m > \lg^k n$ for large enough n.*

Fact 3 *For any positive constant m and integer $n > 0$, we have that $2^n > n^m$ for large enough n.*

Example 3 $\lg 1 = 0$ *as* $2^0 = 1$. $\lg 2^x = x$. $\lg 2^{x+y} = x + y$.

Example 4 *How much is* $(n^{1/\lg n})$*?*

As we don't know the answer, let $x = n^{1/\lg n}$. We take logarithms of both sides of this equality. We get $\lg x = (1/\lg n)\ \lg n = 1$ by rule (13.d), ie $\lg x = 1$. We then take powers of two for both sides i.e. $\lg x = 1$ implies that $2^{\lg x} = 2^1$. The left hand side is x by the definition of the logarithm base two, i.e. $x = 2$. Since $x = n^{1/\lg n}$, we have that $n^{1/\lg n} = 2$. □

Example 5 *For any integer* $n > 0$ *and constant* $k > 0$*, show that for large enough* n $2^n > n^k$*, ie show that there is a constant* n_0 *such that for all* $n > n_0$*, we have that* $2^n > n^k$*.*

Proof (read the proof if you want to brush up on your calculus skills).
Step 1 We intend to show that $2^n > n^k$ for large enough n. We first take logarithms base two of both sides. It then suffices to show that $\lg 2^n > \lg n^k$ that is, $n > k \lg n$. By taking logarithm we reduced our problem to a simple problem.
Step 2 Before proving our claim or the equivalent $n > k \lg n$ that involves a logarithm base two we are going to prove Proposition 2, ie to prove that $n > k \log n$ that involves a natural logarithm. In the proof of Proposition 2 we shall need the following result established in Proposition 1 below.

Proposition 2 *For any real* $m > 0$*, it is* $m^2 > 3 \log m$*.*

Proof (Proposition 1). Take $g(m) = m^2 - 3 \log m$. The first derivative $g'(m) = 2m - 3/m$. A value of m that makes $2m - 3/m = 0$ is $m = \sqrt{1.5} \approx 1.2247$. The second derivative $g''(m) = 2 + 3/m^2$ is always positive for any m, ie $g(m)$ exhibits its minimum for $m = 1.2247$. The minimum value is $g(1.2247) = 1.5 - 0.877 \approx 0.623$, ie $g(m)$ is always positive as its minimum value is more than zero. Therefore $g(m) > 0$ for all $m > 0$, ie $m^2 > 3 \log m$. □
Step 3 We use Proposition 1 to prove Proposition 2 below. Note that by showing Proposition 2 we come close to our final target of proving $n > k \lg n$.

Proposition 3 *For any integer* $n > 0$ *and constant* m*, for large enough* n *we have* $n > m \log n$ *(notice that we have* log *not* lg *here). Large enough* n *would mean that* $n > m^3$*.*

Proof (Proposition 2). Let $f(n) = n - m \log n$. The first derivative $f'(n) = 1 - m/n$ which is zero for $n = m$. We also observe that $f''(n) = m/n^2$ which is positive for $n = m$.

Therefore the minimum value of $f(n)$ is for $n = m$ and it is $f(m) = m - m \log m$. For larger values of n, $f(n)$ grows larger (it is an increasing function) and becomes zero for $n - m \log n = 0$, ie when $n/\log n = m$. Therefore for values of n such that $n/\log n > m$, $f(n)$ is always **positive**.

Note that m is a constant. Let us pick an $n_0 = m^3$. As m is a constant, m^3 is a constant and therefore n_0 is also constant. We note that $n_0/\log n_0 = m^3/(3 \log m)$. But $m^3/(3 \log m) > m$ as this is equivalent to $m^2 > 3 \log m$ of Proposition 1. Therefore $f(n_0)$ is positive and for all $n > n_0$, $f(n)$ is also positive as $n > n_0 > m$.

Therefore we conclude that there exists a constant $n_0 = m^3$ such that for all $n > n_0$, $f(n)$ is positive, ie $f(n) = n - m \log n > 0$, ie $n > m \log n$ for all $n > n_0$.

Note. Our objective was to find some constant n_0, **NOT** the **best possible value of** n_0.
Step 4. We have thus proved that for any constant m and any integer $n > n_0 = m^3$, $n > m \log n$. Our objective is to show that for any constant k, $n > k \lg n$, for large enough n.

As $n > m \log n$ holds for any constant m, we choose constant m so that $m = k/\log 2$, which is also constant. We substitute this for m in $n > m \log n$ and we get $n > (k/\log 2) \log n = k(\log n/\log 2)$. But from an earlier identity on logarithms we have that $\log n/\log 2 = \lg n$, ie $n > k \lg n$.
Therefore for large enough $n > n_0 = m^3 = (k/\log 2)^3$, we have that $n > k \lg n$ and our claim is proved.

As $n > k \lg n$, then $2^n > 2^{k \lg n}$ as well, ie $2^n > n^k$ as well. □

Note. The proof above may seem complicated and it is so. It tells us how we can prove inequalities that involve logarithms, exponentials and polynomials in one inequality. You may read or skip the proof as long as you remember for the remainder of this course the three facts outlined above.

3 Short Notes on Asymptotic Notation

- **NEVER FORGET THAT** $O(f(n))$ is not a function, it is a SET. Therefore a relation of the form $1 \le O(n)$ is nonsense as you compare a constant function (1) to a set using $\le$ a symbol that we use to compare numbers.
- ω vs o? Take the limit of the two functions $f(n)/g(n)$. If the limit is 0, then $f(n) = o(g(n))$. If it is ∞, then $f(n) = \omega(g(n))$.
- If little o is true then O is also true; if $f(n) = o(g(n))$, then $f(n) = O(g(n))$. Prove it.
- If little ω is true then Ω is also true; if $f(n) = \omega(g(n)$, then $f(n) = \Omega(g(n))$. Prove it.

We review the following relationships (n is positive).

- Remember $2^{\lg n} = n$ and $a^{\lg n} = n^{\lg a}$ and $n^{1/\lg n} = 2$.
- $n = \omega(\lg n)$, $n = \omega(\lg^2 n) = \omega((\lg n)^2)$. Note that $\lg^2 n$ means $(\lg n)^2$.
- In general $n^k > \lg^l n$ for any constant k, l and large enough n. It is also true that $n^k = \omega(\lg^l n)$.
- In general $2^n > n^k$ for any constant k and large enough n. It is also true that $2^n = \omega(n^k)$.
- $n! > n^k$ for any constant k and large enough n.
- $n! \approx (n/e)^n \sqrt{2\pi n}$ (Stirling's approximation formula for the factorial). It is also true that $\lg(n!) = O(n \lg n)$.
- Again note that $n^{1/\lg n} = 2$. $n^{1/\lg n}$ seems **to grow fast** but it does not! It is a constant, the constant 2.

Example 6 *Which of* $a_0 + a_1 n + a_2 n^2 + a_3 n^3$ *and* n^2 *is asymptotically larger, where* $a_i > 0$ *for all* i*?*

Proof. Consider $a_0 + a_1 n + a_2 n^2 + a_3 n^3$. As all a_i are positive, then $a_0 > 0$ and $a_1 n > 0$ and $a_2 n^2 > 0$ and thus $a_0 + a_1 n + a_2 n^2 + a_3 n^3 > a_3 n^3$.

Hint. When we intend to prove $f(n) = \Omega(g(n))$, it sometimes helps to find a lower bound $h(n)$ for $f(n)$ ie one such that $f(n) \ge h(n)$ and then show that $h(n) = \Omega(g(n))$. In our case a lower bound for $a_0 + a_1 n + a_2 n^2 + a_3 n^3$ is $a_3 n^3$.

We now show that our lower bound $a_3 n^3$ is $\Omega(n^2)$. As $a_3 > 0$, it is obvious that $a_3 n^3 > 1 \cdot n^2$ for any $n > 1/a_3$ (note that $a_3 > 0$ DOES NOT MEAN THAT $a_3 > 1$, as a_3 is real and not necessarily an integer).
Therefore for $c = 1$ and $n_0 = 1/a_3$ we have shown that $a_0 + a_1 n + a_2 n^2 + a_3 n^3 = \Omega(n^2)$. □

Example 7 *Which of the two functions is asymptotically larger* $a_0 + a_1 n + a_2 n^2 + a_3 n^3$ *or* n^4*, where* $a_i > 0$ *for all* i*?*

Hint. When we intend to prove $f(n) = O(g(n))$, as is the case here, it sometimes helps to find an upper bound $h(n)$ for $f(n)$ ie one such that $f(n) \le h(n)$ and then show that $h(n) = O(g(n))$.
Since in $a_0 + a_1 n + a_2 n^2 + a_3 n^3$ all a_i are positive we take the maximum of all a_i and we call it A. Then we have that $a_i < A$ for all i. Also, $An^i < An^3$ for $i \le 3$. Then

$$a_0 + a_1 n + a_2 n^2 + a_3 n^3 \le A + An + An^2 + An^3 \le An^3 + An^3 + An^3 + An^3 = 4An^3$$

Finally $4An^3 \le n^4$ for all $n > 4A$.
We have shown that

$$a_0 + a_1 n + a_2 n^2 + a_3 n^3 \le 4An^3 \le 1 \cdot n^4$$

for all $n \ge n_0 = 4A$, where A is the maximum of a_0, a_1, a_2, a_3. As all a_i are constant, so is $4A$. Therefore the constants in the O definition are $c = 1$ and $n_0 = 4A$, where $A = \max\{a_0, a_1, a_2, a_3\}$. □