Examples

Example 1 *Solve the recurrence $T(n) = 2T(n/2) + n$ using the substitution (a.k.a. guess-and-check) method. Implicit assumption: $T(n)$ is nonnegative for all n.*

Proof. We observe that no boundary condition is given; we can thus assume T(constant) = some-constant.

Guess a solution.

We guess $T(n) \le cn \lg n$ for all $n_0 \le n$ where c, n_0 are positive constants. This is equivalent to showing that $T(n) = O(n \lg n)$.

We are going to show that the guessed solution is the solution to the recurrence.

We shall prove our claim by using induction. In the inductive proof we shall delay the proof of the BASE CASE until the very end.

Base Case. Proof at the end (let us assume that base case is true for the remainder).

Induction Hypothesis. We assume that the guessed solution is true for all integer values less than n ie that for all $i \le n-1$ we have that

$$T(i) \le ci \lg i \;\text{ for all }\; n_0 \le i < n-1,$$

(Here we renamed the variable in $T(.)$ to i from n to avoid confusion).

Inductive Step.

In the inductive step we show the correctness of our guess for $i = n$ ie for the next largest value of i (by the induction hypothesis the inequality is true for $i \le n-1$).

$$T(i) \le ci \lg i \;\text{ for }\; i = n$$

which is equivalent to showing that

$$T(n) \le cn \lg n$$

Step 1.

As $n/2 < n$, the induction hypothesis applies to $i = n/2$. Therefore,

$$T(n/2) \;\le\; c(n/2) \lg (n/2) \tag{1}$$

Step 2.

In order to prove the induction step we use the only piece of information that we have available for $T(n)$. **This is the recurrence relation**. The inequality below is a result of the induction hypothesis

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\le 2(c(n/2) \lg (n/2)) + n \\ &= 2(c(n/2)(\lg n - 1)) + n \\ &= cn \lg n - cn + n \end{aligned}$$

Step 3.

In order to show that $T(n) \leq cn \lg n$, the last expression $cn \lg n - cn + n$ must be less than or equal to $cn \lg n$. This is so provided that $c \geq 1$. (a **condition on c is thus established**).

Therefore for $c \geq 1$ we have that

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\leq cn \lg n - cn + n \\ &\leq cn \lg n \end{aligned}$$

We have thus shown the inductive step. This completes the induction, ie we have proved that $T(n) \leq cn \lg n$ for any $c \geq 1$.

Base Case Proof. As $\lg 0$ is not defined, we don't define a base case for $T(0)$. The base case is defined for an $n = n_0$, where n_0 is some constant. For $n = n_0$, the recurrence $T(n_0) = 2T(n_0/2) + n_0$ and the claim we want to establish $T(n_0) \leq cn_0 \lg n_0$ must hold.

We first attempt to prove the base case for an $n_0 = 1$. To show that $T(1) \leq c \cdot 1 \cdot \lg 1$, as $c \cdot 1 \cdot \lg 1 = 0$, we must have that $T(1) \leq 0$. As $T(1) \geq 0$ ($T(n)$ is assumed to be non-negative), then we must have $T(1) = 0$ as the boundary condition.

Conclusion 1 (Case $T(1) = 0$)

A solution to the recurrence $T(n) = 2T(n/2) + n$, $T(1) = 0$ is $T(n) \leq cn \lg n$ for any $c \geq 1$ and $n \geq 1$.

Let us assume that we are given the boundary condition $T(1) = 3$.

Then, from the previous discussion $T(1) = 3$ and $T(1) \leq 0$ must BOTH hold, a contradiction. This means that a base case cannot be established for $n_0 = 1$. The next largest value for n_0 is 2.

We thus try the next available integer value $n = 2$. From the recurrence we have that $T(2) = 2T(1) + 2 = 2 \cdot 3 + 2 = 8$, as $T(1) = 3$. Then $T(2) < c \cdot 2 \cdot \lg 2 = 2c$ is true provided that $8 \leq 2c$, ie $c \geq 4$.

Conclusion 2 (Case $T(1) = 3$)

A solution to the recurrence $T(n) = 2T(n/2) + n$, $T(1) = 3$ is $T(n) \leq cn \lg n$ for any $c \geq 4$ and $n \geq 2$.

□

We now solve the same recurrence using the iteration (a.k.a recursion tree) method, a frequently occurring exam problem. We change the boundary condition from the one used in class.

Example 2 *Solve the recurrence $T(n) = 2T(n/2) + n$, $T(2) = 5$ using the iteration/recursion tree method.*

Proof.

We rename variables in the recurrence relation substituting i for n (it doesn't matter whether we have i or n, but the discussion below will be become less confusing).

$$T(i) = 2T(i/2) + i$$

Substituting $n/2$ for i we get.

$$T(n/2) = 2T((n/2)/2) + n/2 = 2T(n/2^2) + n/2$$

Substituting $n/4$ for i we get.

$$T(n/2^2) = T(n/4) = 2T((n/4)/2) + n/4 = 2T(n/2^3) + n/4$$

Similarly we can substitute $n/2^3, n/2^4, \ldots, n/2^i, \ldots, n/2^{lgn} = 1$ for i and get similarly stated recurrences.

We use all the derived recurrences to expand $T(n)$ below. The objective is to sum the equalities and observe in the expansion sequence some symmetries and repetitions for the purpose of combining as many terms as possible to derive a closed form solution for $T(n)$.

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2(2T(n/2^2) + n/2) + n \\
&= 2^2T(n/2^2) + 2 \cdot n/2 + n \\
&= 2^2(2T(n/2^3) + n/2^2) + 2 \cdot n/2 + n \\
&= 2^3T(n/2^3) + 2^2(n/2^2) + 2 \cdot n/2 + n \\
&= 2^3T(n/2^3) + n + n + n \\
&= 2^3T(n/2^3) + 3n \\
&= \ldots \\
&= 2^iT(n/2^i) + i \cdot n
\end{aligned}
$$

From the boundary condition for $m = 2$, which is $T(2) = 5$, we decide when to stop the expansion. The expansion ends so that $n/2^i = 2$ and then $T(n/2^i) = T(2) = 5$. We solve for i by taking logarithms base two of both sides. Then we get that $\lg n - i = 1$ ie $i = \lg n - 1$.

Then, for $i = \lg n - 1$, we get that.

$$
\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2^iT(n/2^i) + i \cdot n \\
&= 2^{\lg n - 1}T(n/2^{\lg n - 1}) + (\lg n - 1) \cdot n \\
&= (n/2)T(2) + (\lg n - 1) \cdot n \\
&= (n/2)5 + (\lg n - 1) \cdot n \\
&= (3n/2) + n \lg n
\end{aligned}
$$

Thus we have obtained the following solution to the recurrence: $T(n) = 3n/2 + n \lg n$.

Checking the solution (THIS IS NOT PART OF A SOLUTION).

How can we be sure of the correctness of this solution? It suffices to show that the derived solution (a) validates the recurrence, and (b) is consistent with $T(2)$.

We start with (b).

$$T(2) = 3 \cdot 2/2 + 2 \lg 2 = 3 + 2 = 5 = T(2).$$

It is obvious that the boundary condition is consistent with the solution.

We proceed to showing (a). We obtained the solution

$$T(n) = 3n/2 + n \lg n.$$

Substituting $n/2$ for n we get that

$$T(n/2) = 3n/4 + (n/2) \lg n/2 = 3n/4 + (n/2)(\lg n - 1) = n/4 + (n/2) \lg n.$$

We start from the right hand side of the recurrence using the preceding equality.

$$2T(n/2) + n = 2(n/4 + n/2 \lg n) + n = 3n/2 + n \lg n.$$

The last term is $T(n)$. We have thus proved that for $T(n) = 3n/2 + n \lg n$, we have that $T(n) = 2T(n/2) + n$, ie our solution satisfies both the recurrence and the boundary condition, ie it is indeed a solution to the recurrence. $\square$.

Example 3 *Solve the recurrence* $T(n) = 8T(n/2) + n$ *using the iteration/recursion tree method. Assume that* $T(1) = 5$.

Proof.

$$
\begin{aligned}
T(n) &= 8T(n/2) + n \\
&= 8(8T(n/2^2) + n/2) + n \\
&= 8^2 T(n/2^2) + 8n/2 + n \\
&= 8^2 T(n/2^2) + (8/2)^1 n + (8/2)^0 n \\
&= 8^2 (8T(n/2^3) + n/2^2) + (8/2)^1 n + (8/2)^0 n \\
&= 8^3 T(n/2^3) + 8^2 n/2^2 + (8/2)^1 n + (8/2)^0 n \\
&= 8^3 T(n/2^3) + (8/2)^2 n + (8/2)^1 n + (8/2)^0 n \\
&= \ldots \\
&= 8^i T(n/2^i) + (8/2)^{i-1} n + \ldots + (8/2)^1 n + (8/2)^0 n
\end{aligned}
$$

Again the boundary case is $T(1) = 5$. We set $n/2^i = 1$, ie $i = \lg n$. Then for $i = \lg n$, $T(n/2^i) = T(1) = 5$. We therefore get for $i = \lg n$.

$$
\begin{aligned}
T(n) &= 8T(n/2) + n \\
&= 8^i T(n/2^i) + (8/2)^{i-1} n + \ldots + (8/2)^1 n + (8/2)^0 n \\
&= 8^{\lg n} T(n/2^{\lg n}) + \frac{(8/2)^{\lg n} - 1}{8/2 - 1} \cdot n \\
&= 2^{3 \lg n} T(1) + \frac{(4)^{\lg n} - 1}{3} \cdot n \\
&= n^3 T(1) + \frac{(4)^{\lg n} - 1}{3} \cdot n \\
&= 5n^3 + \frac{n^2 - 1}{3} n
\end{aligned}
$$

To verify our calculations we observe that $T(1) = 5 + (1-1)/3 = 5$ and

$$
\begin{aligned}
T(n) &= 8T(n/2) + n \\
&= 8(5(n/2)^3 + \frac{(n/2)^2 - 1}{3} n) + n \\
&= 5n^3 + \frac{n^2 - 1}{3} n \\
&= T(n),
\end{aligned}
$$

ie the recurrence is verified.