CIS 786 Homework 3 Solution Outline

Problem 1. (60 points)

Separate response will be given.

Problem 2. (20 points)

We use Valiant's lower bound argument.

Let us have n keys whose maximum MAX_n we must establish in two rounds using p processors. Let us assume that $p \geq n$, since otherwise we can use extra processors for free.

The number of parallel comparisons that can be performed in the first round is p. We form a graph with vertices the keys and edges the comparisons established. There is in the formed graph an independendent set of size at least $P = n^2/(2p+n)$ by Turan's theorem. We can force a second round on the algorithm by fixing these P keys so that they become candidates for the maximum.

In the second round concentrate on these P keys only. A number of p parallel comparisons can determine the Maximum of the P keys (and of course of the original n keys) as long as $p \geq P^2 \geq \binom{P}{2}$ (we use the "CRCW" or the 1-round algorithm described in class). We thus have

$$\begin{aligned}
p &\geq P^2 \geq \binom{P}{2} \\
p &\geq \left(\frac{n^2}{2p+n}\right)^2 \\
p(2p+n)^2 &\geq n^4 \\
p(2p+p)^2 \geq p(2p+n)^2 &\geq n^4 \\
p(2p+p)^2 &\geq n^4 \\
9p^3 &\geq n^4 \\
p &= \Omega\left(n^{4/3}\right).
\end{aligned}$$

Problem 3. (20 points)

We start with the description in the notes and then, we continue it.

Consider n keys. Split them into $n/3$ groups of 3 keys each. For each group we can detemine the MAX in $3(3-1)/2 = 3$ comparisons using 3 procs per group. Total number of processors used is $n/3 \cdot 3 = n$. Thus we are left with determining the max of $n/3$ keys.

Take the $n/3$ Maxima, and split them into groups of 7. We have $n/(3 \cdot 7)$ groups of 7 keys. Each group requires $7(7-1)/2 = 21$ processors to find the MAX of the group. Total number of processors used is $n/21 \cdot 21 = n$, and we can afford to use that many processors. Thus after the second second it suffices to find the max of $n/21$ keys to determine the MAX of the original n keys.

How do we split the $n/21$ keys next? What is the pattern?

Say that at some point we end up having n/s keys. We split them into groups of t keys so that $t(t-1)/2$ comparisons/processors are assigned per group to determine the group maximum in one step(round).

This would give a total of assigned processors equal to $(n/(st))t(t-1)/2$ which must be at most n (number of available processors). For this to hold, if we do the math we end up with $t = 2s+1$. Thus we should split the n/s keys into $n/(s(2s+1))$ groups of $2s+1$ keys. Or for $n/21$ keys, into groups of 43 keys in the following iteration.

What is the pattern? The denominator s in the next step become $s(2s+1)$ or at least $2^1 \cdot s^{2^1}$.

In the next step, it becomes $2(2s^2)^2 = 2^{1+2}s^{2^2}$. In the following step, it becomes $2 \cdot 2^{2+4}s^8 = 2^{1+2+4}s^{2^3}$.

In the i-th step it will become $2^{1+2+2^2+\ldots+2^{i-1}}s^{2^i} = 2^{2^i-1}s^{2^i}$.

This denominator becomes greather than n, i.e. the number of groups is 1 or less and thus MAX can be determined uniquely for i about $\lg\lg n$.