

1 Insertion in red-black trees

Insertion of a node z in an r-b tree is similar to insertion in a BST tree.

z **becomes the root.** If node z is inserted into an empty tree, we color z BLACK, and make z the root of the tree. Otherwise, the tree is not empty and,

z **is not the root.** We perform the standard BST-Insert operations and color z red.

Possible Problem. When we color z red, if the parent pz of z is also red, we have a problem. Note that in that case the grandparent gz of z must be black (or pz could not have been red). Towards this we need to apply a function $FIX(z)$ recursively to fix the RED color of z . (If z becomes the root, fixing the root is straightforward!)

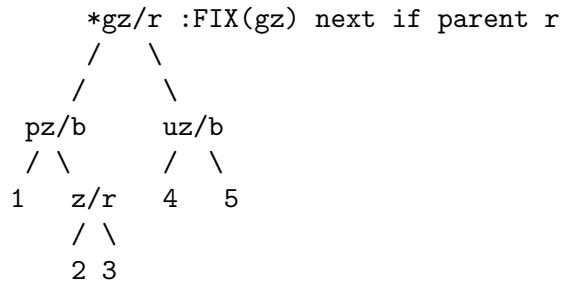
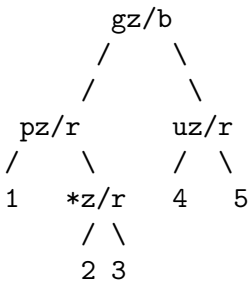
Case Description. Call cases XYc . X is based on the $pz:gz$, Y on the $z:pz$ and c on the $color(uz)$ relationship, where uz is the sibling of pz . Thus X,Y can be R or L and c can be r or b.

1.1 Case 1: LLr, LRr and RRr, RLr

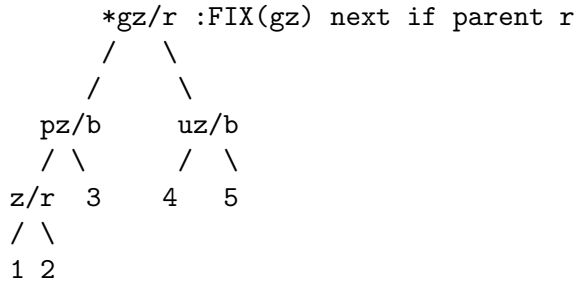
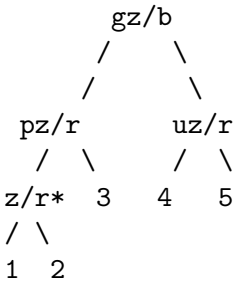
Case 1. The first case involves the Insertion subcases **LRr and LLr** which are shown. Cases **RRr and RLr** are not shown but are symmetric. These cases require node recolorings only. Note that if gz is the root its color cannot change; this causes an increase to the blackheight of descendant nodes. FIX may cause a total of $O(\lg n)$ recursive calls higher in the tree.

// LRr, LLr shown (RRr, RLr symmetric and not shown)

Case 1a: LRr :: gz b \rightarrow r; gz and $p(gz)$ may become red; Call $Fix(gz)$ next.



Case 1b: LLr :: Same as before (gz : b \rightarrow r)

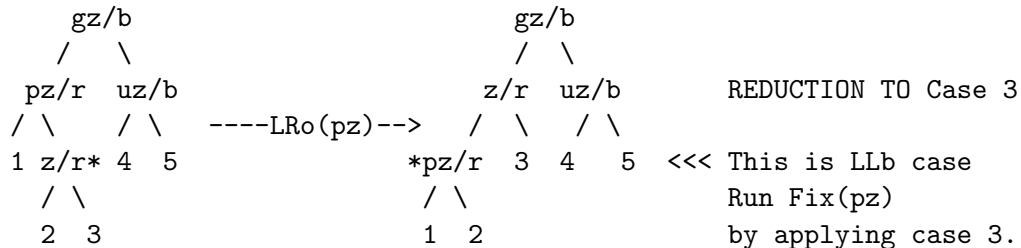


1.2 Case 2: LRb, and Case 3: LLb

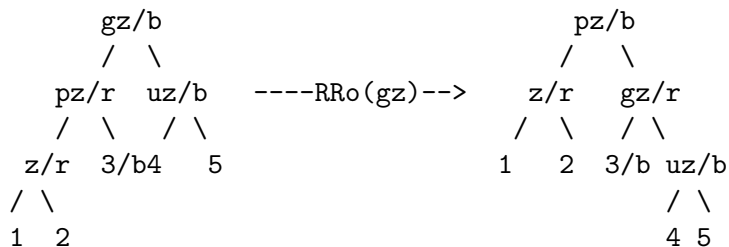
Case 2 covers the case of LRb, and Case 3 the case of LLb. Case 2 is ALWAYS FOLLOWED by Case 3. RLb and RRb are symmetric(not shown).

***** A star (*) shows the node on which FIX is run.

Case 2:LRb is reduced to Case 3:LLb and resolved.



Case 3:LLb



After the single rotation is performed there can be no way that there are two consecutive RED nodes in a path from the root to pz (root of the subtree in Case 3). Therefore after a single rotation we are done.

Conclusion. Insertion requires $O(\lg n)$ recolorings (Case 1) and $O(1)$ rotations (Cases 2 and 3).

2 Deletion in red-black trees

Deletion in an r-b tree is similar to Deletion in a BST tree. When we perform Delete(z), a node is spliced out; this node is called x . If z has no or one child, then x is z otherwise x is the successor (or the predecessor) of z . In any of these three cases we call y the only child of x (if x has no children, then y is the NULL only child of x reminding ourselves that in an rb-tree there is only one NULL node), and py the new parent of y after the spliceout, which was previously the parent of x .

- a. $(x,y,py)=(r,b,b)$. If x is red, then y must be black and py must be black or otherwise x should have been black. Splicing out x causes no violations whatsoever.
- b. $(x,y,py)=(b,?,?)$ Splicing out a black x causes a violation of RB3. We distinguish the following subcases.
 - b1. $(x,y,py)=(b,r,?)$. If y is red, we recolor y black and the violation is resolved.
 - b2. $(x,y,py)=(b,b,?)$ and y is root. If y is black and y becomes the root of the tree, no RB3 violation occurs, because all the paths from the root y will have black height one less.
 - b3. $(x,y,py)=(b,b,?)$ and y not root. If y is black but not the root, we have a violation of RB3 that can not be resolved immediately. We “transfer” the BLACK color of x to y by coloring y DOUBLE-BLACK. We then need to fix y by calling FIXDELETE(y), i.e. a node y of FIXDELETE is a node “colored” BLACK twice.

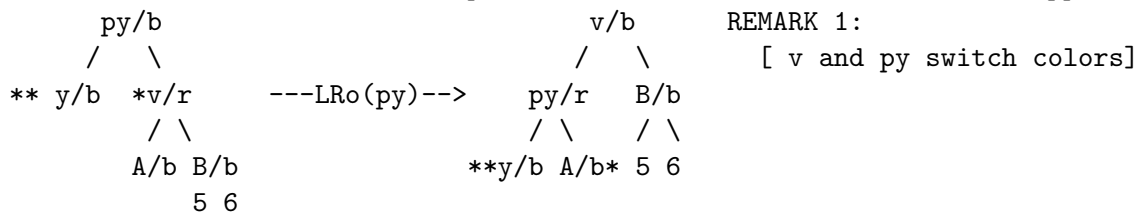
Reminder: delete(z) \rightarrow spliceout(x) \rightarrow FixDelete(y).

Case Description. All cases are labeled Xcn, where X denotes the orientation of $y:py$, c the color of the sibling u of y and n the number of red children of u (i.e. sibling of y).

2.1 Cases 1 and 2

Comment: Lr1 and Lr2 are not possible; a red node CANNOT have red children.

Case 1: Lr0 : A left rotation is performed and then Case 2a or 3 or 4 applies.



Case 2. Lb0

Subcase 2a. If py is r (Case 1) color py with b and color v with r and stop

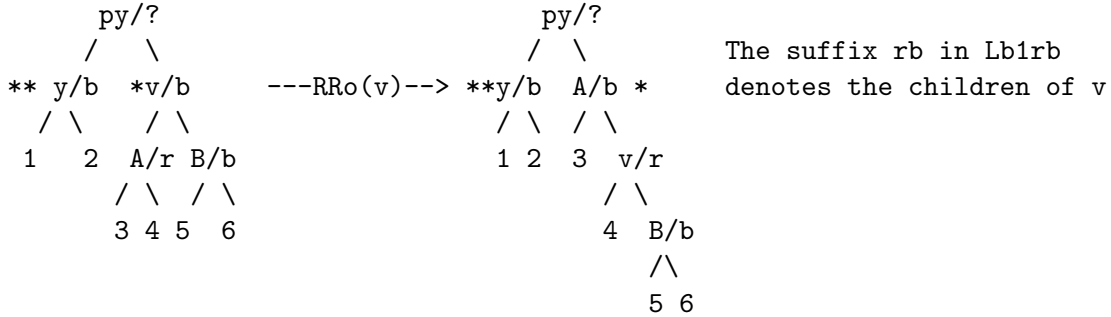


Subcase 2b, If py is b FIXDELETE(py) py plays the role of y and py also carries the extra black inherited by y due to the x splice out.

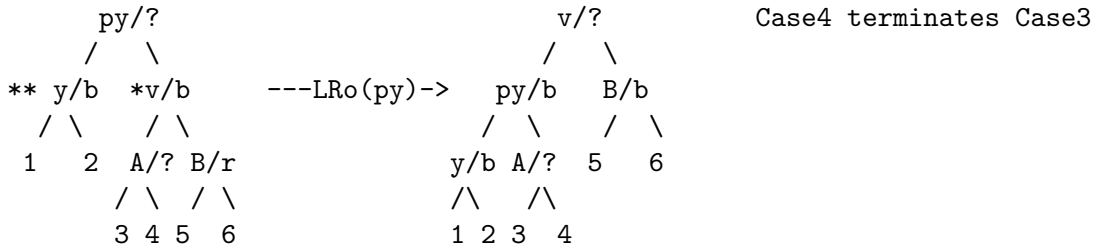


2.2 Cases 3 and 4

Case 3: Lb1rb : Is transformed into case 4 immediately A,v exchanging colors.



Case 4: Lb2 and Lb1br (there is neither Lb1bb=Lb0 neither Lb1rr=Lb2)



Running time is $O(\lg n)$ as well. Cases 1, 2a, 3, 4 terminate in $O(1)$ time, Case 2b advances (moves towards the top) one level every time it is executed, and the height of the RB tree is $O(\lg n)$.