1 Insertion in red-black trees

Insertion of a node \( z \) in an \( r-b \) tree is similar to insertion in a BST tree.

- \( z \) becomes the root. If node \( z \) is inserted into an empty tree, we color \( z \) BLACK, and make \( z \) the root of the tree. Otherwise, the tree is not empty and,

- \( z \) is not the root. We perform the standard BST-Insert operations and color \( z \) red.

Possible Problem. When we color \( z \) red, if the parent \( p_z \) of \( z \) is also red, we have a problem. Note that in that case the grandparent \( g_z \) of \( z \) must be black (or \( p_z \) could not have been red). Towards this we need to apply a function \( \text{FIX}(z) \) recursively to fix the RED color of \( z \). (If \( z \) becomes the root, fixing the root is straightforward!)

Case Description. Call cases \( XYc \). \( X \) is based on the \( p_z: g_z \), \( Y \) on the \( z:p_z \) and \( c \) on the \( \text{color}(uz) \) relationship, where \( uz \) is the sibling of \( p_z \). Thus \( X,Y \) can be \( R \) or \( L \) and \( c \) can be \( r \) or \( b \).

1.1 Case 1: LLr, LRr and RRr, RLr

Case 1. The first case involves the Insertion subcases LRr and LLr which are shown. Cases RRr and RLr are not shown but are symmetric. These cases require node recolorings only. Note that if \( g_z \) is the root its color cannot change; this causes an increase to the blackheight of descendant nodes. \( \text{FIX} \) may cause a total of \( O(\lg n) \) recursive calls higher in the tree.

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// LRr, LLr shown (RRr, RLr symmetric and not shown)

Case 1a: LRr :: \( g_z \rightarrow b \rightarrow r \); \( g_z \) and \( p(g_z) \) may become red; Call \( \text{Fix}(g_z) \) next.

\[
\begin{array}{c}
gz/b \\
/ \ \ \\
/ \ \\
pz/r \\
/ \ \\
/ \\
z/r* \\
/ \\
2 3 \\
\end{array}
\]

\[
\begin{array}{c}
*gz/r : \text{FIX}(gz) \text{ next if parent } r \\
/ \\
/ \\
pz/b \\
/ \\
/ \\
z/r 4 5 \\
/ \\
1 \\
\end{array}
\]

Case 1b: LLr :: Same as before (\( g_z: b \rightarrow r \))

\[
\begin{array}{c}
gz/b \\
/ \ \\
/ \\
pz/r \\
/ \\
/ \\
z/r* \\
/ \\
1 2 \\
\end{array}
\]

\[
\begin{array}{c}
*gz/r : \text{FIX}(gz) \text{ next if parent } r \\
/ \\
/ \\
pz/b \\
/ \\
/ \\
z/r 3 4 5 \\
/ \\
2 3 \\
\end{array}
\]
1.2 Case 2: LRb, and Case 3: LLb

Case 2 covers the case of LRb, and Case 3 the case of LLb. Case 2 is ALWAYS FOLLOWED by Case 3. RLb and RRb are symmetric (not shown).

***** A star (*) shows the node on which FIX is run.
Case 2: LRb is reduced to Case 3: LLb and resolved.

\[
\begin{align*}
  &gz/b & gz/b \\
  &/ \ &/ \\
  &pz/r \, uz/b & z/r \, uz/b \quad REDUCTION \ TO \ Case \ 3 \\
  &/ \ / \ ----LRo(pz)--> / \ / \\
  &1 \, z/r* \ 4 \ 5 & *pz/r \ 3 \ 4 \ 5 \ \text{<< This is LLb case} \\
  &/ \ / \ &/ \ \text{Run Fix(pz)} \\
  &2 \ 3 & 1 \ 2 \ \text{by applying case 3.}
\end{align*}
\]

Case 3: LLb

\[
\begin{align*}
  &gz/b & pz/b \\
  &/ \ &/ \\
  &pz/r \, uz/b ----RRo(gz)--> z/r \, gz/r \\
  &/ \ / \ &/ \ / \\
  &z/r \ 3/b4 \ 5 & 1 \ 2 \ 3/b \ uz/b \\
  &/ \ &/ \\
  &1 \ 2 & 4 \ 5
\end{align*}
\]

After the single rotation is performed there can be no way that there are two consecutive RED nodes in a path from the root to pz (root of the subtree in Case 3). Therefore after a single rotation we are done.

**Conclusion.** Insertion requires $O(\log n)$ recolorings (Case 1) and $O(1)$ rotations (Cases 2 and 3).
2 Deletion in red-black trees

Deletion in an r-b tree is similar to Deletion in a BST tree. When we perform Delete(z), a node is spliced out; this node is called x. If z has no or one child, then x is z otherwise x is the successor (or the predecessor) of z. In any of these three cases we call y the only child of x (if x has no children, then y is the NULL only child of x reminding ourselves that in an rb-tree there is only one NULL node), and py the new parent of y after the spliceout, which was previously the parent of x.

a. \((x,y,py)=(r,b,b)\). If x is red, then y must be black and \(p(y)\) must be black or otherwise x should have been black. Splicing out x causes no violations whatsoever.

b. \((x,y,py)=(b,?,?)\) Splicing out a black x causes a violation of RB3. We distinguish the following subcases.

b1. \((x,y,py)=(b,r,?)\). If y is red, we recolor y black and the violation is resolved.

b2. \((x,y,py)=(b,b,?)\) and y is root. If y is black and y becomes the root of the tree, no RB3 violation occurs, because all the paths from the root y will have black height one less.

b3. \((x,y,py)=(b,b,?)\) and y not root. If y is black but not the root, we have a violation of RB3 that cannot be resolved immediately. We “transfer” the BLACK color of x to y by coloring y DOUBLE-BLACK. We then need to fix y by calling FIXDELETE(y), i.e. a node y of FIXDELETE is a node “colored” BLACK twice.

Reminder: delete(z) → spliceout(x) → FixDelete(y).

Case Description. All cases are labeled \(Xcn\), where \(X\) denotes the orientation of \(y:py\), \(c\) the color of the sibling \(u\) of \(y\) and \(n\) the number of red children of \(u\) (i.e. sibling of \(y\)).

2.1 Cases 1 and 2

Comment: Lr1 and Lr2 are not possible; a red node CANNOT have red children.

Case 1: Lr0 : A left rotation is performed and then Case 2a or 3 or 4 applies.

Case 2. Lb0

Subcase 2a. If py is r (Case 1) color py with b and color v with r and stop

Subcase 2b. If py is b FIXDELETE(py) py plays the role of y and py also carries the extra black inherited by y due to the x splice out.
2.2 Cases 3 and 4

Case 3: Lb1rb : Is transformed into case 4 immediately A,v exchanging colors.

```
py/?                                 py/?
/ \                                 / \  The suffix rb in Lb1rb
** y/b *v/b  ---RRo(v)--> **y/b A/b * denotes the children of v
/ \  / \                                    / \ / \
1 2 A/r B/b 1 2 3 v/r
/ \ / \                                  / \ / \  \
3 4 5 6    4 B/b                        1 2 3 4
/ \                                                   / \  
5 6
```

Case 4: Lb2 and Lb1br (there is neither Lb1bb=Lb0 neither Lb1rr=Lb2)

```
py/?                                      v/?
/ \                                      / \\ Case4 terminates Case3
** y/b *v/b  ---LRo(py)--> py/b B/b
/ \  / \                                / \ / \
1 2 A/? B/r y/b A/? 5 6
/ \ / \                              / \ / \  \
3 4 5 6    1 2 3 4
```

Running time is \(O(\lg n)\) as well. Cases 1, 2a, 3, 4 terminate in \(O(1)\) time, Case 2b advances (moves towards the top) one level every time it is executed, and the height of the RB tree is \(O(\lg n)\).