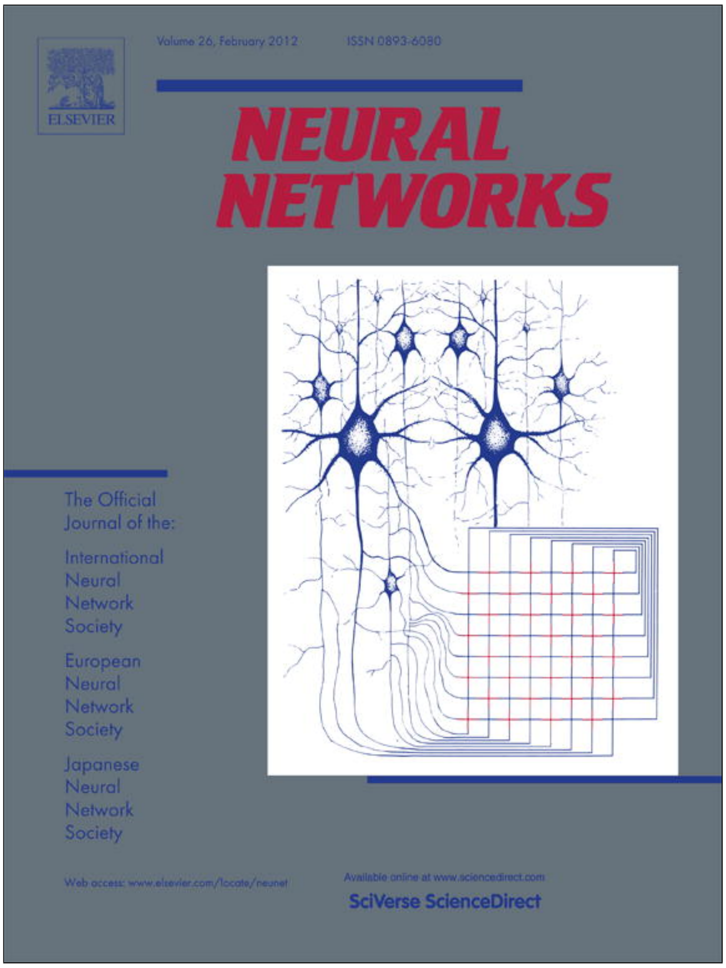
Volume 26, February 2012
ISSN 0893-6080
ELSEVIER
NEURAL NETWORKS
The Official Journal of the:
International Neural Network Society
European Neural Network Society
Japanese Neural Network Society
Web access: www.elsevier.com/locate/neunet
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Neural Networks 26 (2012) 110–117

Contents lists available at SciVerse ScienceDirect

Neural Networks

journal homepage: www.elsevier.com/locate/neunet

Theoretic design of differential minimax controllers for stochastic cellular neural networks

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ARTICLE INFO

Article history:
Received 25 November 2010
Received in revised form 5 September 2011
Accepted 6 September 2011

Keywords:
Stochastic cellular neural networks
Differential minimax game
Hamilton–Jacobi–Bellman equation
Lyapunov function
Stochastic stability

ABSTRACT

This paper presents a theoretical design of how a minimax equilibrium of differential game is achieved in stochastic cellular neural networks. In order to realize the equilibrium, two opposing players are selected for the model of stochastic cellular neural networks. One is the vector of external inputs and the other is the vector of internal noises. The design procedure follows the nonlinear H infinity optimal control methodology to accomplish the best rational stabilization in probability for stochastic cellular neural networks, and to attenuate noises to a predefined level with stability margins. Three numerical examples are given to demonstrate the effectiveness of the proposed approach.

1. Introduction

The past two decades have witnessed enormous advances in engineering and in computer science to build artificial computational systems (Werbos, 2009), among which cellular neural networks are being implemented by using microprocessor chips and have been readily applied to many scientific and engineering fields such as, pattern recognition, image processing, DNA micro-array analysis, satellite data transmissions, etc (Arena, Bucolo, Fortuna, & Occhipinti, 2002; Arik, 2002; Aziz & Lara, 2007; Chua & Roska, 2005). Many applications of them require a well-defined solution for all possible initial conditions under different circumstances. From a mathematical point of view, this signifies that the network should have a unique equilibrium point, which is both stable and globally attractive (Van den Driessche, Wu, & Zou, 2001). Moreover, when the cellular neural networks are implemented by very large-scale integrated (VLSI) circuits, time delays arise in the processing of signal transmission and information storage among the neurons, which lead to more complicated dynamics such as oscillation phenomenon or network instability (Cheng, 2009). Therefore, the stability of cellular neural networks has become an important topic in the past decade (see for example, Arik (2002), Cao (2003), He, Wu, and She (2006), Huang, Huang, and Liu (2005), Kwon, Park, and Lee (2008), Liao and Wang (2000, 2003), Xia, Xia, and Liu (2008), Xu, Lam, Ho, and Zou (2005) and references therein), especially the stability of delayed cellular neural networks. However, all the cellular neural networks discussed above are either autonomous networks without considering inputs or having a constant input vector. Because the model of cellular neural networks (in any form) usually involves an external input that can be used as a control input, it makes good sense to develop some controllers to achieve the stabilization. Furthermore, the aforementioned studies primarily focused on deterministic cellular neural networks. In the mathematical models of these aforementioned networks, they do not consider the noise process that is fraught with signal transmission particularly in biological systems.

On the other hand, Haykin (Haykin, 1994) indicated that the synaptic transmission is a noise process in real nervous systems. Therefore, as Werbos (Werbos, 2009) pointed out that in order to develop mathematical neural network specifications which have dual uses as models of intelligence in the brain, and as highly effective artificial intelligent systems when implemented in computers and chips, we must consider the stochastic environment. Unfortunately, with regard to the analysis of stochastic cellular neural networks, there has been little work in the literature until the very recent years (Huang & Yang, 2009; Lu & Ma, 2008; Wan & Zhou, 2011). Moreover, all the stochastic cellular neural networks discussed by those aforementioned publications are autonomous cellular neural networks without considering external inputs. Hence,

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0893-6080/$ – see front matter
doi:10.1016/j.neunet.2011.09.003

it is important to analytically explore the characteristics of controllability and stabilization by using external inputs (control inputs) for time-delay cellular neural networks under the influence of stochastic perturbation.

As an extension of our previous study (Liu, Torres, Patel, & Wang, 2008; Liu, Wang, & Schurz, 2009), this paper presents a theoretic design of differential minimax controllers for stochastic time-delay cellular neural networks to achieve both the best rational stabilization in probability under an optimal control strategy, and to attenuate noise to a predefined level within stability margins. By applying the theory of differential game to the networks, the approach is developed by considering the vector of external inputs as a player and the vector of internal noises as the opposing player. Therefore, a minimax equilibrium can be achieved by properly controlling stochastic cellular neural networks. The rest of the paper is organized as follows. In Section 2, we present the problem formulation and mathematical preliminaries. In Section 3, we detail the theoretical results. In Section 4, we demonstrate the performance of our design with three numerical examples. Finally, the conclusion of the paper is given in Section 5.

2. Problem formulation and preliminaries

In this paper, we consider the following model of stochastic time-delay cellular neural networks

$$dx_i(t) = \left(-\lambda x_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(t-\tau)) + u_i\right) dt + dw_i(t) \quad (1)$$

where $i = 1, 2, \ldots, n$. Mathematically, this can be described in the following Ito-type matrix–vector compact form

$$dx(t) = (-Ax(t) + Bg(x(t)) + Cg(x(t-\tau)) + u)dt + dw \quad (2)$$

where $x(t) \in R^n$ is the state of the stochastic time-delay cellular neural network, $u \in R^n$ is the input, $A = \mathrm{diag}(\lambda, \ldots, \lambda) = \lambda I \in R^{n\times n}$ and $\lambda > 0$, $g(x(t)) = [g_1(x_1(t)), \ldots, g_j(x_j(t)), \ldots, g_n(x_n(t))]^T \in R^n$ is a vector function, and $g_j(x_j(t))$ is a sigmoidal function (scalar one) that models the nonlinear input–output activations of the neurons, $B \in R^{n\times n}$ and $C \in R^{n\times n}$ are weight matrices, $\tau \in R^+$ is the time delay, w is an n-dimensional independent Wiener process with incremental covariance $\sum(t)\sum(t)^T dt$, i.e., $E\{dwdw^T\} = \sum(t)\sum(t)^T dt$ where $\sum(t)$ is an unknown bounded function taking values in the set of nonnegative definite matrices.

Our design goal is to develop a feedback controller for the stochastic cellular neural networks modeled by Eq. (2) to achieve the best rational stabilization in probability and to attenuate noises to a predefined level with stability margins under an optimal control strategy.

Before we derive the main analytical results in the next section, it is necessary to introduce the following definitions.

Definition 2.1. The norm $\|x\|$ of a vector x is the Euclidean norm. If A is a matrix, then $\|A\|$ denotes the Frobenius matrix norm, defined as $\|A\| = (T_r\{A^TA\})^{1/2}$, where $T_r\{.\}$ denotes the trace of a matrix. If $f : R^n \to R^n$ is a vector field and $V : R^n \to R$ is a scalar function, the notation $L_f V$ is used to denote $\frac{\partial V}{\partial x} f(x) = \frac{\partial V}{\partial x_1} f_1(x) + \cdots + \frac{\partial V}{\partial x_i} f_i(x) + \cdots + \frac{\partial V}{\partial x_n} f_n(x)$.

Definition 2.2. Let us define a nonnegative bounded function

$$\Delta(t) = \Sigma(t)\Sigma(t)^T \in R^{n\times n} \quad (3)$$

which will be used as a player opposing an external control in order to solve the game problem addressed in this paper.

Definition 2.3. The activation function $g_j(x_j(t))$ has the following properties:

(i) $g_j(0) = 0$.
(ii) The function $g_j : R \to R$ is monotonically increasing and globally Lipschitz with constant $k_j > 0$, i.e. $|g_j(x_i) - g_j(y_i)| \le k_j|x_i - y_i|$, $\forall x_i \in R$, $\forall y_i \in R$, with $|\cdot|$ the respective absolute value.

3. Main results

We develop our theoretic design based on the concepts of differential minimax game (Basar & Bernhard, 1995) together with inverse optimality (Freeman & Kokotovic, 1996), which is one of the most promising methods in the area of modeling biologically-inspired neural networks (Todorov, 2006; Werbos, 2009). Following the technique of inverse optimality (Freeman & Kokotovic, 1996; Krstic & Deng, 1998), we first need to find a stabilizing control and then modify it to optimize a meaningful cost functional.

Theorem 1. *Consider the system of stochastic time-delay cellular neural network modeled by Eq. (2), if we choose the control as*

$$u = -(2 + \|B\|^2k^2 + 2\|C\|^2k^2)x(t), \quad (4)$$

then the system achieves noise-to-state stabilization.

Proof. Let us rewrite the system of the stochastic cellular neural networks represented by Eq. (2) as

$$dx(t) = (-Ax(t) + Bg(x(t)) + Cg(x(t-\tau)))dt + udt + dw. \quad (5)$$

Now we define a Lyapunov function V as

$$V = \frac{1}{2}x(t)^Tx(t) + \int_{t-\tau}^{t} ((Cg(x(s)))^T(Cg(x(s))))ds. \quad (6)$$

The infinitesimal generator given by Blythe, Mao, and Liao (2001) is

$$\begin{aligned} LV &= x(t)^T(-Ax(t) + Bg(x(t)) + Cg(x(t-\tau))) + x(t)^Tu \\ &\quad + \frac{1}{2}T_r\left\{\sum(t)^T\sum(t)\right\} + (Cg(x(t)))^T(Cg(x(t))) \\ &\quad - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \\ &= -\lambda x(t)^Tx(t) + x(t)^TBg(x(t)) + x(t)^T \\ &\quad \times Cg(x(t-\tau)) + x(t)^Tu + \frac{1}{2}T_r\left\{\sum(t)^T\sum(t)\right\} \\ &\quad + (Cg(x(t)))^T(Cg(x(t))) \\ &\quad - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))). \end{aligned} \quad (7)$$

Let us apply the following Young's Inequality to both the second term $x(t)^TBg(x(t))$ and the third term $x(t)^TCg(x(t(t-\tau)))$ in (7),

$$x^Ty \le \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2} \quad (8)$$

in which x and y are two vectors.

We obtain

$$\begin{aligned} x(t)^TBg(x(t)) &\le \frac{1}{2}x(t)^Tx(t) + \frac{1}{2}\|Bg(x(t))\|^2 \\ &\le \frac{1}{2}x(t)^Tx(t) + \frac{1}{2}\|B\|^2\|g(x(t))\|^2 \end{aligned} \quad (9)$$

and

$$x(t)^TCg(x(t-\tau)) \le \frac{1}{2}x(t)^Tx(t) + \frac{1}{2}(Cg(x(x-\tau)))^T(Cg(x(x-\tau))). \quad (10)$$

From Definition 2.3, we have

$$|g_j(x_j)| \le k_j|x_j|, \quad \forall x_j \in R,\ j = 1, 2, \ldots, n \tag{11}$$

and

$$\|g(x(t))\|^2 \le (Kx(t))^T(Kx(t)) = x(t)^T K^2 x(t) \le k^2 x(t)^T x(t) \tag{12}$$

where $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$ and $k = \max\{k_j\},\ j = 1, \ldots, n$.

From Eq. (9), we gain

$$x(t)^T Bg(x(t)) \le \frac{1}{2}x(t)^T x(t) + \frac{1}{2}\|B\|^2 k^2 x(t)^T x(t) = x(t)^T \times \left(\frac{1 + \|B\|^2 k^2}{2}\right) x(t). \tag{13}$$

In addition

$$(Cg(x(t)))^T(Cg(x(t))) = \|Cg(x(t))\|^2 \le \|C\|^2 k^2 x(t)^T x(t). \tag{14}$$

Substitute Eqs. (10), (13) and (14) into Eq. (7), we get

$$\begin{aligned} LV \le\ & -\lambda x(t)^T x(t) + x(t)^T\left(\frac{1 + \|B\|^2 k^2}{2}\right)x(t) + \frac{1}{2}x(t)^T x(t) \\ & + \frac{1}{2}(Cg(x(x-\tau)))^T(Cg(x(x-\tau))) \\ & + x(t)^T u + \frac{1}{2}T_r\left\{\sum(t)^T \sum(t)\right\} \\ & + (Cg(x(t)))^T(Cg(x(t))) - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \\ \le\ & -\lambda x(t)^T x(t) + x(t)^T\left(\frac{1 + \|B\|^2 k^2}{2}\right)x(t) \\ & + \frac{1}{2}x(t)^T x(t) + x(t)^T u + \frac{1}{2}T_r\left\{\sum(t)^T \sum(t)\right\} \\ & + \|C\|^2\|g(x(t))\|^2 - \frac{1}{2}(Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \\ \le\ & -\lambda x(t)^T x(t) + x(t)^T\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)x(t) \\ & + x(t)^T u + \frac{1}{2}T_r\left\{\sum(t)^T \sum(t)\right\} \\ & - \frac{1}{2}(Cg(x(t-\tau)))^T(Cg(x(t-\tau))). \end{aligned} \tag{15}$$

Because of $(Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \ge 0$,

we finally achieve

$$LV \le -\lambda x(t)^T x(t) + x(t)^T\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)x(t) + x(t)^T u + \frac{1}{2}T_r\left\{\sum(t)^T \sum(t)\right\}. \tag{16}$$

With the control of Eq. (4), the inequality above becomes

$$LV \le -\lambda x(t)^T x(t) + \frac{1}{2}T_r\left\{\sum(t)^T \sum(t)\right\} - x(t)^T \times \left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)x(t). \tag{17}$$

Let us define

$$\alpha(|x|) = x(t)^T\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)x(t) = \left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)\|x(t)\|^2. \tag{18}$$

Thus, we get

$$LV \le -x(t)^T\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right)x(t) = -\alpha(|x|) \tag{19}$$

whenever $\|x(t)\| \ge \frac{\left(\frac{1}{2}T_r\{\sum(t)^T \sum(t)\}\right)^{1/2}}{\sqrt{\lambda}} = \frac{\left(\frac{1}{2}T_r(\Delta(t))\right)^{1/2}}{\sqrt{\lambda}}$.

By the definition of noise-to-state stability (Krstic & Deng, 1998, p. 78, Definition 4.1) and Theorem 4.2 of the reference Krstic and Deng (1998, p. 78), we conclude that the system of Eq. (2) is noise-to-state stabilization with the control of Eq. (4). This completes the proof. □

Remark 3.1. Although the Lyapunov theory, the most successful and widely used tool, was invented a century ago, there are still no systematic methods to obtain Lyapunov functions for general nonlinear systems. Therefore, it remains a challenging task to find a Lyapunov function and stabilize a nonlinear system (Primbs, Nevistic, & Doyle, 1999).

Next let us discuss how to achieve both the best rational stabilization in probability for stochastic cellular neural networks, and attenuate noises to a predefined level with stability margins under an optimal control strategy. Here, we apply the concepts of differential minimax game (Basar & Bernhard, 1995) and inverse optimality (Freeman & Kokotovic, 1996). Regarding the system of Eq. (2), we consider the control $u(t)$ as a player to oppose the noise function $\Delta(t)$, which is another player defined in Definition 2.2, in the game of controlling the system modeled by Eq. (2).

Consider the following general stochastic nonlinear system that is affined in the noise w and control u

$$dx = f(x)dt + g_1(x)dw + g_2(x)udt. \tag{20}$$

If there exists a positive optimal value function $V(x)$, which satisfies the following Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial t} + L_f V + \frac{1}{4}\gamma^{-2}\left\|g_1^T(x)\frac{\partial^2 V}{\partial x^2}g_1(x)\right\|^2 - \frac{1}{4}L_{g_2}Vr^{-1}(x)L_{g_2}^T V + q(x) = 0 \tag{21}$$

then

$$u^*(x) = -\frac{1}{2}r^{-1}(x)Lg_2^T V \tag{22}$$

is an optimal stabilizing control which minimizes the cost functional

$$J(u, \Delta) = \lim_{t\to\infty} E\left[V(x(t)) + \int_0^t \times \left(q(x) + u^T r(x)u - \frac{\gamma^2}{4}\|\Delta\|^2\right)ds\right] \tag{23}$$

where $\gamma > 0$ is a predefined design parameter, both $q(x) \ge 0$ and $r(x) > 0$ for all x, and the worst case $\Delta^*(t)$ is

$$\Delta^*(t) = \frac{1}{\gamma^2}\left(g_1^T(x)\frac{\partial^2 V}{\partial x^2}g_1(x)\right). \tag{24}$$

Remark 3.2. It is still an open problem to find the solution of Hamilton–Jacobi–Bellman equation (21) for the general stochastic nonlinear system (20).

We now consider the Lyapunov function V as optimal value function and substitute it into Eq. (21), which yields the next equation

$$(Cg(x(t)))^T(Cg(x(t))) - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) - \lambda x(t)^T x(t) + x(t)^T Bg(x(t)) + x(t)^T Cg(x(t-\tau)) + \frac{1}{4}\gamma^{-2}n - \frac{1}{4}x(t)^T r^{-1}(x(t))x(t) + q(x(t)) = 0. \quad (25)$$

Let us define a new control that is a modification of Eq. (4)

$$u = -c\left(\frac{2+\|B\|^2k^2+2\|C\|^2k^2}{2}+\frac{2}{c}\right)x(t) = -\frac{1}{2}c\left(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c}\right)x(t) \quad (26)$$

where $c > 2$ is a constant.

We then choose the function $r(x(t))$ as

$$r(x(t)) = c^{-1}(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c})^{-1} \quad (27)$$

and from Eq. (25) the function $q(x(t))$ is given by

$$q(x(t)) = \lambda x(t)^T x(t) + \frac{c}{4}(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c})x(t)^T x(t) - \frac{1}{4}\gamma^{-2}n - x(t)^T Bg(x(t)) - x(t)^T Cg(x(t-\tau)) + (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) - (Cg(x(t)))^T(Cg(x(t))). \quad (28)$$

Now let us assume

$$\|x(t)\| \geq \max\left\{\frac{\sqrt{n}}{2\gamma}, \frac{\gamma}{2\sqrt{\lambda}}\|\Delta\|\right\}. \quad (29)$$

With the assumption above, we can now derive the following theorem.

Theorem 2. *For the stochastic cellular neural network described by Eq. (2), there exist a positive-definite function $q(x(t))$ Eq. (28) and a strictly positive function $r(x(t))$ Eq. (27), such that the feedback control law*

$$u = u^* = -\frac{1}{2}r^{-1}(x(t))x(t) \quad (30)$$

achieves an optimal noise-to-state stabilization with respect to a meaningful cost functional

$$J(u,\Delta) = \lim_{t\to\infty} E\left[V(x(t)) + \int_0^t \left(q(x(s)) + u^T r(x(s))u - \frac{\gamma^2}{4}\|\Delta(s)\|^2\right)ds\right] \quad (31)$$

for the worst case $\Delta(t)$

$$\Delta(t) = \Delta^*(t) = \frac{1}{\gamma^2}I. \quad (32)$$

Furthermore, a minimax equilibrium (u^*, Δ^*) is achieved.

Proof. *Step* 1: By considering the Lyapunov function candidate V, the infinitesimal generator of the stochastic differential equation (2) is

$$LV = -\lambda x(t)^T x(t) + x(t)^T Bg(x(t)) + x(t)^T Cg(x(t-\tau)) + x(t)^T u + \frac{1}{2}T_r\left\{\sum(t)^T\sum(t)\right\} + (Cg(x(t)))^T \times (Cg(x(t))) - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))). \quad (33)$$

The substitution of the control law Eq. (30) into LV yields

$$LV = -\lambda x(t)^T x(t) + x(t)^T g(x(t)) + x(t)^T Cg(x(t-\tau)) - x(t)^T\left(\frac{1}{2}c\left(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c}\right)x(t)\right) + \frac{1}{2}Tr\{\Delta(t)\} + (Cg(x(t)))^T(Cg(x(t))) - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))). \quad (34)$$

From Eqs. (10), (13), (14) and (29), we obtain

$$\begin{aligned}
LV &\leq -\lambda x(t)^T x(t) + x(t)^T\left(\frac{1+\|B\|^2k^2}{2}\right)x(t) \\
&\quad + \frac{1}{2}x(t)^T x(t) + \frac{1}{2}(Cg(x(x-\tau)))^T(Cg(x(x-\tau))) - x(t)^T \\
&\quad \times \left(\frac{1}{2}c\left(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c}\right)x(t)\right) + \frac{1}{2}Tr\{\Delta(t)\} \\
&\quad + (Cg(x(t)))^T(Cg(x(t))) - (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \\
&\leq -\lambda x(t)^T x(t) - \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right) \\
&\quad \times x(t)^T x(t) - x(t)^T x(t) + \frac{1}{2}T_r\{\Delta(t)\} \\
&\quad - \frac{1}{2}(Cg(x(t-\tau)))^T(Cg(x(t-\tau))) \\
&\leq -\lambda x(t)^T x(t) - \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right) \\
&\quad \times x(t)^T x(t) - \frac{n}{4\gamma^2} + \frac{1}{2}T_r\{\Delta(t)\} \\
&= -\lambda x(t)^T x(t) - \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right) \\
&\quad \times x(t)^T x(t) - T_r\left\{\left(\frac{1}{2\gamma}I - \frac{\gamma}{2}\Delta(t)\right)^T\left(\frac{1}{2\gamma}I - \frac{\gamma}{2}\Delta(t)\right)\right\} \\
&\quad + \frac{\gamma^2}{4}\|\Delta(t)\|^2 \\
&\leq -\lambda x^T x + \frac{\gamma^2}{4}\|\Delta(t)\|^2 \\
&\quad - \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right)x(t)^T x(t). \quad (35)
\end{aligned}$$

Let us define

$$\alpha(|x|) = \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right)x(t)^T x(t) = \left(\left(\frac{c-1}{2}\right)(2+\|B\|^2k^2+2\|C\|^2k^2)+1\right)\|x(t)\|^2. \quad (36)$$

Therefore $LV \leq -\alpha(|x|)$ whenever $\|x\| \geq \frac{\gamma}{2\sqrt{\lambda}}\|\Delta(t)\|$.

Same as Theorem 1, we conclude that the system of Eq. (2) achieves noise-to-state stabilization with the control law Eq. (30).

Step 2: Let us consider $q(x(t))$ and $r(x(t))$

By Eq. (28)

$$q(x(t)) = \lambda x(t)^T x(t) + \frac{c}{4}\left(2+\|B\|^2k^2+2\|C\|^2k^2+\frac{4}{c}\right) \times x(t)^T x(t) - \frac{1}{4}\gamma^{-2}n - x(t)^T Bg(x(t)) - x(t)^T \times Cg(x(t-\tau)) + (Cg(x(t-\tau)))^T(Cg(x(t-\tau))) - (Cg(x(t)))^T(Cg(x(t))). \quad (37)$$

Substitute Eqs. (10), (13), (14) and (29) into Eq. (37) above, we get

$$\begin{aligned} q(x(t)) &\geq \lambda x(t)^T x(t) + \frac{c}{4}\left(2 + \|B\|^2 k^2 + 2\|C\|^2 k^2 + \frac{4}{c}\right) \\ &\quad \times x(t)^T x(t) - \frac{1}{4}\gamma^{-2} n - x(t)^T \left(\frac{1 + \|B\|^2 k^2}{2}\right) x(t) \\ &\quad - \frac{1}{2} x(t)^T x(t) - \frac{1}{2}(Cg(x(x-\tau)))^T (Cg(x(x-\tau))) \\ &\quad + (Cg(x(t-\tau)))^T (Cg(x(t-\tau))) - \|C\|^2 k^2 x(t)^T x(t) \\ &= \lambda x(t)^T x(t) + \left(\frac{c-2}{2}\right)\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right) x(t)^T \\ &\quad \times x(t) + x(t)^T x(t) - \frac{1}{4}\gamma^{-2} n \\ &\quad + \frac{1}{2}(Cg(x(t-\tau)))^T (Cg(x(t-\tau))) \\ &\geq \lambda x(t)^T x(t) + \left(\frac{c-2}{2}\right)\left(\frac{2 + \|B\|^2 k^2 + 2\|C\|^2 k^2}{2}\right) \\ &\quad \times x(t)^T x(t) \\ &\geq 0. \end{aligned} \tag{38}$$

Then, $q(x(t))$ is positive definite and radially unbounded.

By Eq. (27)

$$r(x(t)) = c^{-1}\left(2 + \|B\|^2 k^2 + 2\|C\|^2 k^2 + \frac{4}{c}\right)^{-1} \tag{39}$$

it is obvious that $r(x(t)) > 0$.

With the choice of $q(x(t))$ and $r(x(t))$ in Eqs. (28) and (27), LV can be written into the following form

$$\begin{aligned} LV &= -q(x(t)) - u^T r(x(t)) u + \frac{\gamma^2}{4}\|\Delta(t)\|^2 - T_r \\ &\quad \times \left\{\left(\frac{1}{2\gamma} I - \frac{\gamma}{2}\Delta(t)\right)^T \left(\frac{1}{2\gamma} I - \frac{\gamma}{2}\Delta(t)\right)\right\} \\ &\quad + (u - u^*)^T r(x(t))(u - u^*). \end{aligned} \tag{40}$$

According to Dynkin's formula (Oksendal, 2002), we have

$$\begin{aligned} J(u, \Delta) &= \lim_{t\to\infty} E\left[V(x(t)) + \int_0^t \left(q(x(s)) + u^T r(x(s)) u \right.\right. \\ &\quad \left.\left. - \frac{\gamma^2}{4}\|\Delta(s)\|^2\right) ds\right] \\ &= \lim_{t\to\infty} E\left[V(x(0)) + \int_0^t \left(LV + q(x(s)) + u^T r(x(s)) u \right.\right. \\ &\quad \left.\left. - \frac{\gamma^2}{4}\|\Delta(s)\|^2\right) ds\right] \\ &= E[V(x(o))] + \lim_{t\to\infty} E\int_0^t \left[(u - u^*)^T r(x(s))(u - u^*) \right. \\ &\quad \left. - T_r\left\{\left(\frac{1}{2\gamma} I - \frac{\gamma}{2}\Delta(s)\right)^T \left(\frac{1}{2\gamma} I - \frac{\gamma}{2}\Delta(s)\right)\right\}\right] ds. \end{aligned} \tag{41}$$

From the above equation, we know that the optimal control $u = u^*$ is an optimal solution to J Eq. (31) for the worst disturbance $\Delta(t) = \Delta(t)^*$, that is,

$$\min_u \max_\Delta J(u, \Delta) = E[V(x(o))]. \tag{42}$$

Therefore, by considering the control $u(t)$ as a player and the noise covariance $\Delta(t)$ as the opposing player, a minimax equilibrium (u^*, Δ^*) is achieved (Basar & Bernhard, 1995). This completes the proof. □

Remark 3.3. In the society of control engineering, there is a strong motivation for designing optimal systems because such systems automatically have many desirable properties, such as, stability, robustness, reduced sensitivity, etc., (Moylan & Anderson, 1973). Because it is too difficult to solve the Hamilton–Jacobi–Bellman equation, the problem of finding a direct optimal control solution for nonlinear systems remains open. Therefore, the research community resorts to an alternative approach, inverse optimality. On the other hand, recent research results have shown that the inverse optimal control appears to be one of the most promising methods in the area of modeling biologically-inspired neural networks (Todorov, 2006; Werbos, 2009). There are many strong evidences to support that biological movements are optimal. However, the exact cost function that is being optimized in a particular mission is not always clear.

4. Numerical examples

In this section, we consider three examples, with which the networks have different structures and activation functions, to verify the theoretical analysis. To consider all situations from various perspectives, a low level of disturbance noises is applied to Example 1 and a high level of disturbance noises is applied to Example 2. In addition, Example 3 presents a more realistic application experiment, which shows one of applications of time-delay cellular neural networks. Three numerical examples are simulated by using the Matlab/Simulink software under the fourth order Runge–Kutta method with a fixed small step size of 0.005.

Example 1. A stochastic time-delay cellular neural network is given as

$$\begin{aligned} \begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} &= \left(-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2.1 & -0.12 \\ -5.1 & 3.2 \end{bmatrix}\right. \\ &\quad \times \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix} + \begin{bmatrix} -1.6 & -0.1 \\ -0.2 & -2.4 \end{bmatrix} \\ &\quad \left. \times \begin{bmatrix} \tanh(x_1(t-\tau)) \\ \tanh(x_2(t-\tau)) \end{bmatrix}\right) dt + \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt \end{aligned} \tag{43}$$

where $x_1(0) = 0.3, x_2(0) = -3, \lambda = -1, B = \begin{bmatrix} 2.1 & -0.12 \\ -5.1 & 3.2 \end{bmatrix}, C = \begin{bmatrix} -1.6 & -0.1 \\ -0.2 & -2.4 \end{bmatrix}$, activation function $g_j(x_j) = \tanh(x_j)$ $(j = 1, 2)$, $\tau = 1$, and w_1, w_2 are white noises (uniformly random) with the magnitude of $|w_j| = 5$ $(j = 1, 2)$.

Fig. 1 shows the result of phase plane of the network without both the noise and the control, which tells that the original network is actually a chaotic delayed cellular neural network. Figs. 2 and 3 display the results of phase plane and time response of the neural network with the noise ($|w_j| = 5$) but without the control. The system is unstable and chaotic. The effect of the noise in the system can be seen obviously, which further worsen the system. Finally, the control signal Eq. (30) is applied to the system at $t = 50$. The result is given by Fig. 4. One can see that the network is globally asymptotically stable, that is, it achieves the stochastic noise-to-state stabilization.

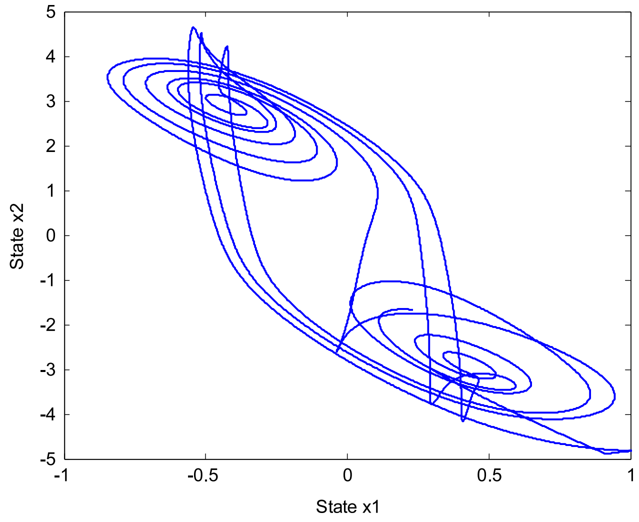

Fig. 1. Phase plane ($u = 0$ and $w_j = 0$ $(j = 1, 2)$).

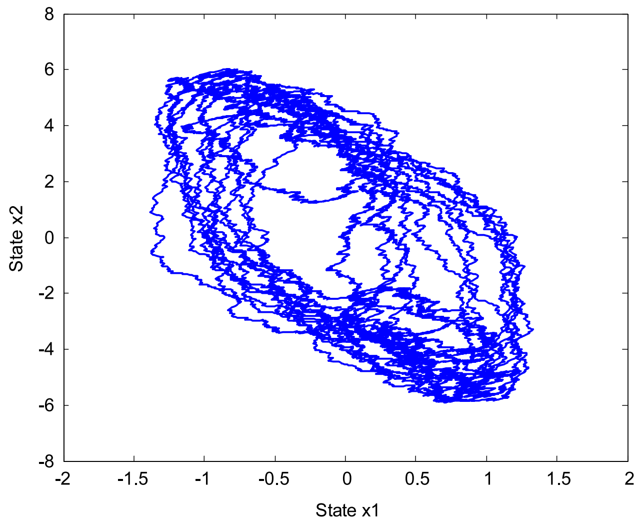

Fig. 2. Phase plane ($u = 0$ and $|w_j| = 5$ $(j = 1, 2)$).

Example 2. Let us consider the following stochastic time-delay cellular neural network

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \left(-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \right. \\ \times \begin{bmatrix} s(x_1(t)) \\ s(x_2(t)) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \left. \begin{bmatrix} s(x_1(t-\tau)) \\ s(x_2(t-\tau)) \end{bmatrix} \right) dt \\ + \begin{bmatrix} dw_1 \\ dw_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} dt \tag{44}$$

where $x_1(0) = 5, x_2(0) = -5, \lambda = -1, B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, activation function $g_j(x_j) = 1/(1+\exp(-2x_j)) - 0.5$ $(j = 1, 2)$, $\tau = 1$, and w_1, w_2 are white noises (uniformly random) with the magnitude of $|w_j| = 20$ $(j = 1, 2)$.

Fig. 5 represents the result of phase plane of the network without both the noise and the control. The system is not globally asymptotically stable. Figs. 6 and 7 display the results of phase plane and time response of the neural network with the noise ($|w_j| = 20$) but without the control. The system is unstable and becomes chaotic. Fig. 8 demonstrates the result of the time responses when the proposed control signal Eq. (30) is inputted at $t = 50$ to the system. It can be seen that the system achieves

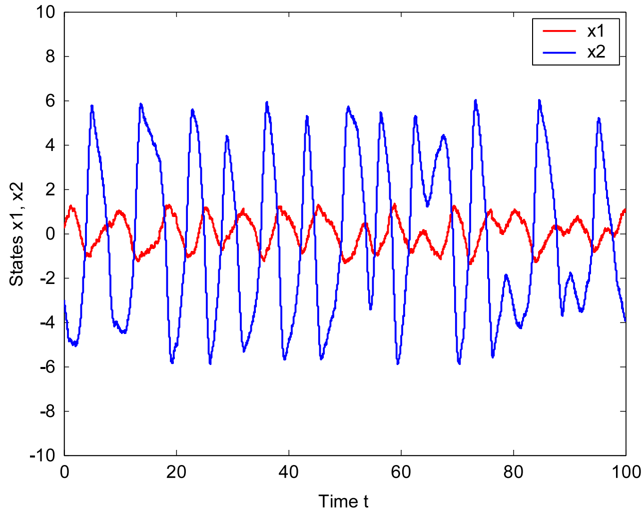

Fig. 3. System response ($u = 0$ and $|w_j| = 5$ $(j = 1, 2)$).

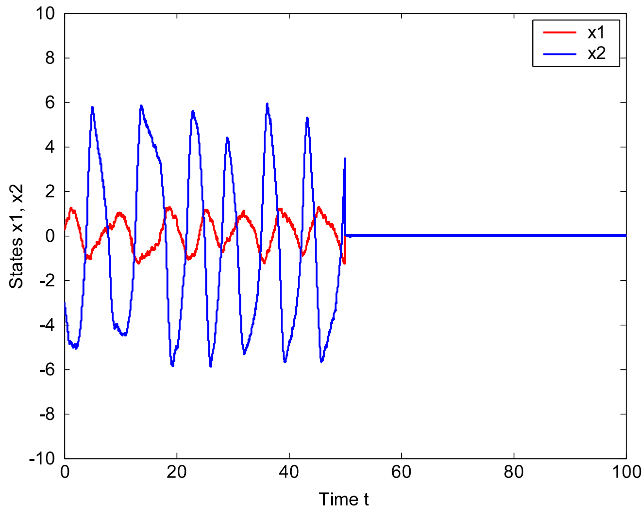

Fig. 4. System response with the control ($u =$ Eq. (30) at $t = 50$).

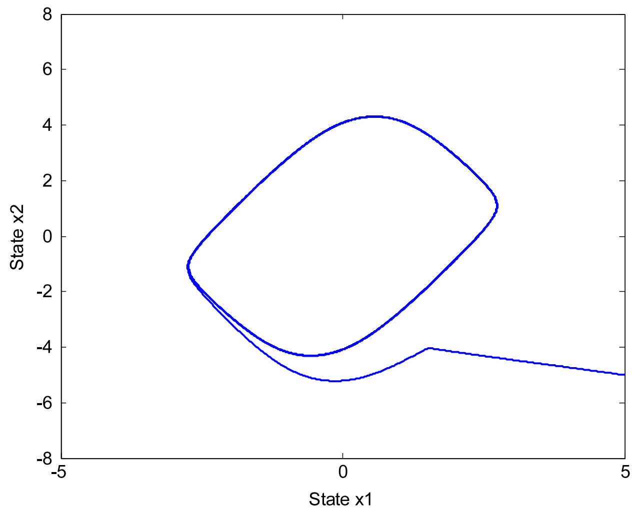

Fig. 5. Phase plane ($u = 0$ and $w_j = 0$ $(j = 1, 2)$).

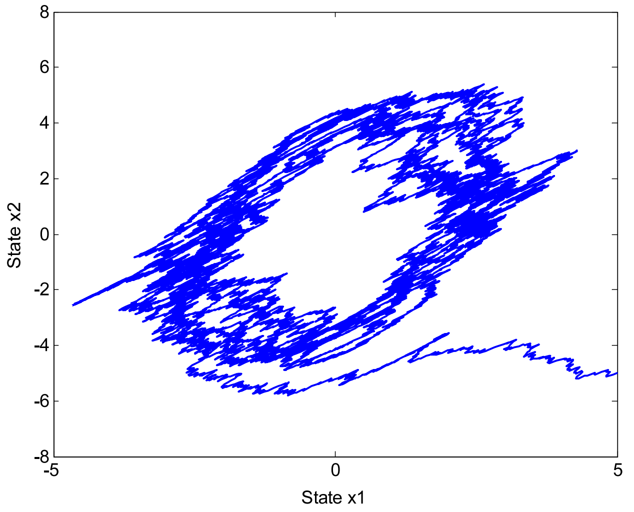

Fig. 6. Phase plane ($u = 0$ and $|w_j| = 20$ ($j = 1, 2$)).

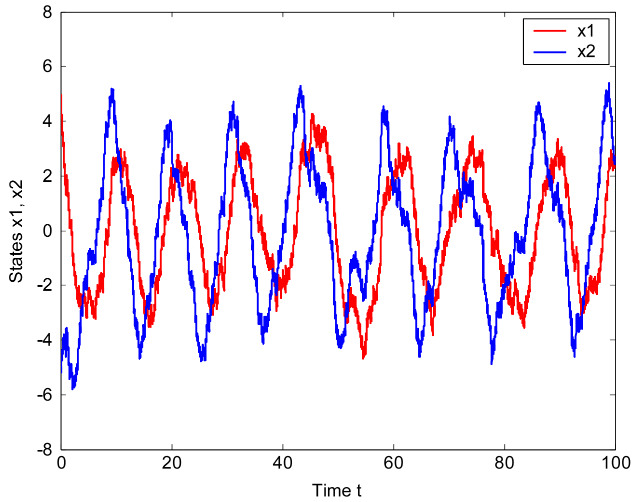

Fig. 7. System response ($u = 0$ and $|w_j| = 20$ ($j = 1, 2$)).

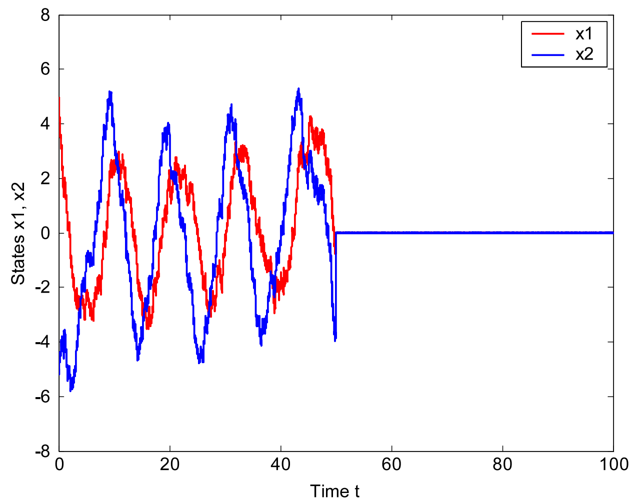

Fig. 8. System response with the control ($u =$ Eq. (30) at $t = 50$).

the expected performance which conforms the aforementioned theoretical analysis in Section 3.

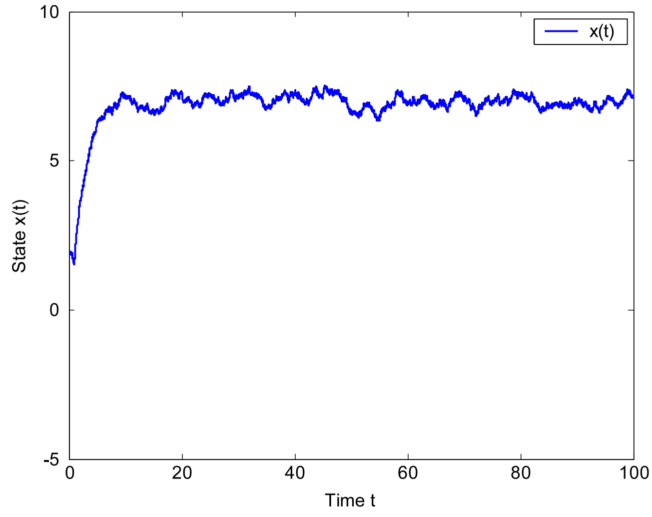

Fig. 9. System Response ($u = 0$ and $|w| = 5$).

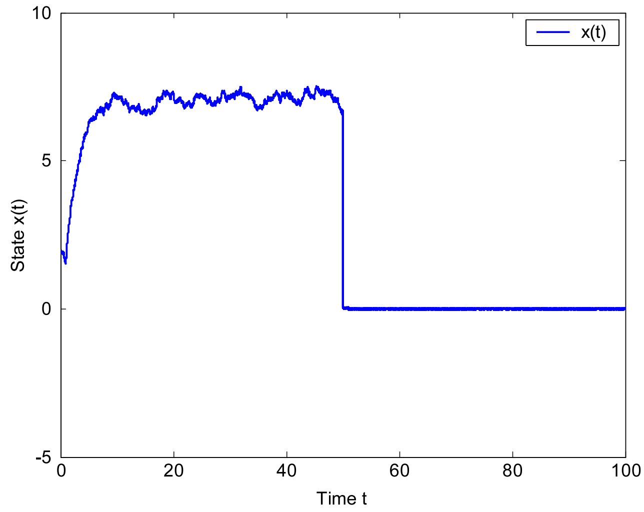

Fig. 10. System response with the Control ($u =$ Eq. (30) at $t = 50$).

Example 3. A time-delay cellular neural network can be used to model a continuous PH neutralization of an acid stream (Zhang, Wang, & Liu, 2008). Therefore, we consider the following stochastic time-delay cellular neural network that is the extension of the application example in Zhang et al. (2008).

$$dx(t) = (-Ax(t) + W \tan h(x(t)) + W_1 \tan h(x(t - \tau)))dt + dw + udt \quad (45)$$

where $x(0) = 2, A = 0.5, W = 0.5, W_1 = 3, \tau = 1$, and w is a white noise (uniformly random) with the magnitude of $|w| = 5$.

Fig. 9 shows the result of time response of the neural network with the noise ($|w| = 5$) but without the control. It is obvious that the system is not globally asymptotically stable. Fig. 10 illustrates the result of the time response when the proposed control signal Eq. (30) is inputted at $t = 50$ to the system. One can see that the network is globally asymptotically stable, that is, it achieves the stochastic noise-to-state stabilization.

5. Conclusions

This paper has presented a new theoretical design for stochastic time-delay cellular neural networks to achieve an optimal noise-to-state stabilization in probability. The proposed approach is developed by using differential minimax game, inverse optimality, Lyapunov technique, and Hamilton–Jacobi–Bellman equation. After considering the vector of external inputs as a player and the vector of internal noises as the opposing player, a minimax equilibrium is achieved in controlling stochastic cellular neural networks. Owing to the difficulty in solving the Hamilton–Jacobi–Bellman equation for nonlinear systems, optimal stabilization seems to be an unachievable goal in feedback design. However, an alternative way has been proposed in this paper to solve the problem and obtain an optimal feedback controller with respect to a meaningful cost functional by using the knowledge of inverse optimality. Simulation results show that the proposed approach turns out to be very effective for different systems under different circumstances. It is believed that the new design presented in this paper would accelerate the applications of stochastic cellular neural networks. Further research is under way to extend our results to other types of neural networks.

Acknowledgments

The authors sincerely thank the Action Editor and the anonymous reviewers for their helpful comments, which have helped them to greatly improve the quality of the paper.

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