

Nonlinear Filtering by Threshold Decomposition

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Abstract—A new threshold decomposition architecture is introduced to implement stack filters. The architecture is also generalized to a new class of nonlinear filters known as *threshold decomposition* (TD) filters which are shown to be equivalent to the class of LI-filters under certain conditions. Another new class of filters known as linear and order-statistic (LOS) filters result from the intersection of the class of TD and LI-filters. Performance comparison among several filters are then presented. It was found that TD is compatible with LI, LOS, and linear filters in suppressing Gaussian noise, and is superior in suppressing salt-and-pepper noise. LOS filters, however, provide a better compromise in performance and complexity.

Index Terms—L-filters, LI-filters, linear and order-statistic filters, nonlinear filters, stack filters, threshold decomposition.

I. INTRODUCTION

LINEAR filters are optimal in eliminating additive white Gaussian noise (AWGN), but in practice, the noise in a channel through which a signal is transmitted is not AWGN; it is not stationary, and it may have unknown characteristics. Therefore, a number of nonlinear filters have been proposed to suppress non-AWGN noise [1]–[5].

Stack filters [1], [6], [7] are a class of sliding-window nonlinear filters characterized by two properties: the threshold decomposition property and the stacking property. They are effective in suppressing impulsive noise, and allow an efficient VLSI implementation. Replacing positive Boolean functions in stack filter by linear operators results in a new class of filters known as *threshold decomposition* (TD) filters, which are more analytically tractable.

LI-filters [2] are another type of nonlinear filters that generalize the order statistic filters (L-filters) [8], [9] and the nonrecursive linear filters (FIR). LI-filters are also effective in recovering signals from non-Gaussian noise, and capable of preserving details.

Though the structure of TD filters and LI-filters are quite different, they still form a common subset—a new type of filters: linear and order-statistic (LOS) filters.

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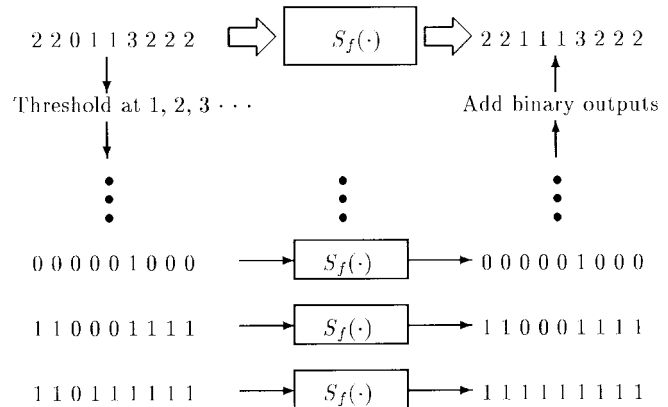


Fig. 1. Stack filter.

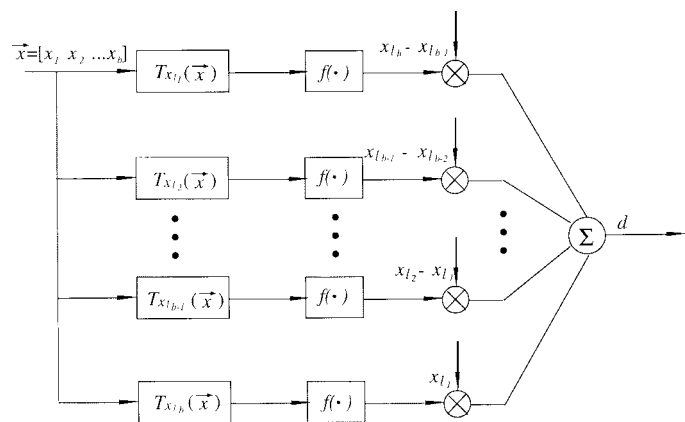


Fig. 2. The new architecture.

II. BACKGROUND

Stack filters is a class of sliding window nonlinear digital filters. Any stack filter can be implemented by the threshold decomposition architecture shown in Fig. 1.

Assume the input $x(n)$ can take on discrete values of $0, 1, \dots, M - 1$, and denote the b samples in the window at time n as

$$\vec{x} = [x_1 \ x_2 \ \dots \ x_b]. \tag{1}$$

The stack filter's output at time n is

$$S(\vec{x}) = \sum_{k=1}^{M-1} f(T_k(\vec{x})) \tag{2}$$

where T_k denotes the thresholding operation. $T_k(\vec{x})$ is a vector with the same size as \vec{x} . $T_k(x_i)$, the i th element of $T_k(\vec{x})$, is one if $x_i \geq k$, and zero, otherwise. f is the positive Boolean function on each level. The index n is omitted for convenience.



Fig. 3. Experimental results. (a) Original Lena image. (b) Original woman1 image. (c) and (d) Images of (a) and (b) corrupted by Gaussian noise.

In the threshold decomposition architecture, the input signal is decomposed via thresholding into a set of $M - 1$ binary signals, and Boolean operation is applied to each of the threshold signals in parallel via Boolean table look-up. The output is the sum of the filtered signals on each level.

It was observed [10] that there are at most b different threshold signals among the $M - 1$ threshold signals $T_k(\vec{x})$, $k = 1, 2, \dots, M - 1$. These different binary signals can be denoted as $T_{x_{l_i}}(\vec{x})$, $i = 1, 2, \dots, b$, where l_i denotes the spatial index of the i th rank sample in the window. In other words, x_{l_i} denotes the i th rank sample. Here, a sample of smaller value is given a smaller rank.

By combining repetitive threshold levels, a new architecture for implementing stack filters is introduced as shown in Fig. 2.

$$S(\vec{x}) = \sum_{i=1}^b f(T_{x_{l_i}}(\vec{x}))(x_{l_i} - x_{l_{i-1}}), \quad x_{l_0} = 0. \quad (3)$$

Note that the number of threshold levels is reduced to b in the new architecture, but extra ranking operation is needed. One distinctive feature of this new architecture is that the threshold decomposition is data-dependent. It leads to the following desirable properties.

Property 2.1: A discrete shift at the input results in a discrete shift at the output. For example, for any integer α ,

$$S(\vec{x} + \alpha\vec{1}) = f(\vec{1})\alpha + S(\vec{x}). \quad (4)$$

If we impose $f(\vec{1}) = 1$, then the filtering operation is shift-invariant, where $\vec{1}$ is a vector of size b with each element equal to one.

Property 2.2: The filtering operation is invariant to discrete scale change of the input, i.e., for any integer β

$$S(\beta\vec{x}) = \beta S(\vec{x}). \quad (5)$$

Note that the above properties hold for general filtering operation on each level.

III. TD FILTERS

In the sequel, we assume different linear operators are used on each level in the new architecture, and the resulting class of filters will be referred to as TD filters. Let the coefficients of the linear operators on the i th level in the new architecture



Fig. 3. (Continued.) Experimental results. (e) and (f) are (c) and (d) filtered by a 3×3 linear filter that is configured from (a) and (c). (g) and (h) are images of (c) and (d) filtered by a 3×3 TD filter that is configured from (a) and (c).

be denoted as \vec{w}_i , then the output of the TD filter is

$$S(\vec{x}) = \sum_{i=1}^b \vec{w}_i T_{x_{l_i}}(\vec{x}) \delta_i \quad (6)$$

where $\vec{w}_i T_{x_{l_i}}(\vec{x})$ denotes the inner product between vector \vec{w}_i and vector $T_{x_{l_i}}(\vec{x})$, and $\delta_i = x_{l_i} - x_{l_{i-1}}$. In general, a TD filter has b^2 coefficients.

Property 3.1: An FIR is a TD filter.

Proof: Any FIR can be regarded as a TD filter which employs the same operator on each level.

Let $x_{i,j} = T_{x_{l_i}}(x_j)$, and denote $w_{i,j}$ as the j th entry of \vec{w}_i . Equation (6) can be rewritten as

$$S(\vec{x}) = \sum_{i=1}^b \sum_{j=1}^b x_{i,j} w_{i,j} \delta_i \quad (7)$$

$$= \sum_{i=1}^b \delta_i \sum_{j=1}^b x_{i,j} w_{i,j}. \quad (8)$$

Since

$$x_{i,l_j} \delta_i = \begin{cases} \delta_i & \text{if } j \geq i \\ 0 & \text{else} \end{cases} \quad (9)$$

we have

$$S(\vec{x}) = \sum_{i=1}^b \delta_i \sum_{j=i}^b w_{i,l_j}. \quad (10)$$

Furthermore

$$\begin{aligned} S(\vec{x}) &= \sum_{i=1}^b (x_{l_i} - x_{l_{i-1}}) \sum_{j=i}^b w_{i,l_j} (x_{l_0} = 0) \\ &= x_{l_b} w_{b,l_b} + x_{l_{b-1}} \left(\sum_{j=b-1}^b w_{b-1,l_j} - w_{b,l_b} \right) \\ &\quad + x_{l_{b-2}} \left(\sum_{j=b-2}^b w_{b-2,l_j} - \sum_{j=b-1}^b w_{b-1,l_j} \right) \\ &\quad + \cdots + x_{l_1} \left(\sum_{j=1}^b w_{1,l_j} - \sum_{j=2}^b w_{2,l_j} \right) \\ &= x_{l_b} w_{b,l_b} + \sum_{i=1}^{b-1} x_{l_i} \left(\sum_{j=i}^b w_{i,l_j} - \sum_{j=i+1}^b w_{i+1,l_j} \right). \end{aligned} \quad (11)$$



Fig. 3. (Continued.) Experimental results. (i) and (j) are images of (c) and (d) filtered by a 3×3 LI-filter that is configured from (a) and (c). (k) and (l) are images of (c) and (d) filtered by a 3×3 LOS filter which is configured from (a) and (c).

The above equation indicates that TD filtering is a linear operation where the weights to a sample depends on both its rank and its spatial location. These operations turn out to be similar to LI filtering [2].

IV. RELATION TO LI-FILTERS

The output of an LI-filter can be expressed as

$$L(\vec{x}) = \sum_{i=1}^b x_{l_i} v_{i, l_i} \quad (12)$$

where v_{i, l_i} is the weight to the i th rank sample at spatial location l_i . The motivation behind the development of LI filtering is to enhance the impulsive noise suppression capability of linear filters. The gain in performance is derived from utilizing rank information of the samples in the window.

Even though TD filters and LI-filters are similar, they are equivalent only when the coefficients satisfy a set of conditions. These conditions are established below.

Property 4.1: An LI-filter is a TD filter iff its coefficients satisfy the following conditions:

$$v_{i-1, j+1} - v_{i-1, j} = v_{i, j+1} - v_{i, j} \quad (13)$$

or equivalently

$$v_{i-1, j} - v_{i, j} = v_{i-1, j+1} - v_{i, j+1} \quad (14)$$

for $j = 1, 2, \dots, b-1$ and $i = 2, 3, \dots, b$.

Proof: Equating (11) and (12), and by successive substitutions, we have

$$w_{b, l_b} = v_{b, l_b} \quad (15)$$

$$\sum_{j=b-1}^b w_{b-1, l_j} = v_{b-1, l_{b-1}} + v_{b, l_b} \quad (16)$$

$$\sum_{j=b-2}^b w_{b-2, l_j} = v_{b-2, l_{b-2}} + v_{b-1, l_{b-1}} + v_{b, l_b} \quad (17)$$



Fig. 3. (Continued.) Experimental results. (m) and (n) are images of (c) and (d) filtered by a 3×3 median filter. (o) and (p) are images of (a) and (b) corrupted by salt-and-pepper noise.

$$\sum_{j=i}^b w_{i,l_j} = v_{i,l_i} + v_{i+1,l_{i+1}} + \dots + v_{b,l_b} \quad (18)$$

$$\sum_{j=2}^b w_{2,l_j} = v_{2,l_2} + v_{3,l_3} + \dots + v_{b,l_b} \quad (19)$$

$$\sum_{j=1}^b w_{1,l_j} = v_{1,l_1} + v_{2,l_2} + \dots + v_{b,l_b}. \quad (20)$$

Note that in the above equations, the left hand side remains the same for any permutation of $(l_i, l_{i+1}, \dots, l_b)$. Hence, there exists a solution $w_{i,j}^s$ for a given $v_{i,j}^s$ only if the right hand side (RHS) remains constant for any permutation of $(l_i, l_{i+1}, \dots, l_b)$.

Given the condition in (18), a swap of any two indices of $(l_i, l_{i+1}, \dots, l_b)$ does not change the value of the RHS. Since any two permutations can be related via successive swaps of two indices, the RHS of (18) is constant for any

permutation of $(l_i, l_{i+1}, \dots, l_b)$. This establishes (13) as a sufficient condition.

In (18), let the subset of indices $(l_{i+2}, l_{i+3}, \dots, l_b)$ be fixed, then (13) is necessary to keep the RHS constant when (l_i, l_{i+1}) is swapped. This establishes (13) as a necessary condition. Hence, we have shown that (13) is a necessary and sufficient condition for an LI-filter to be a TD filter.

Property 4.2: A TD filter is an LI-filter iff its coefficients satisfy the following conditions:

$$w_{i,j} - w_{i+1,j} = w_{i,j+1} - w_{i+1,j+1} \quad (21)$$

or

$$w_{i,j} - w_{i,j+1} = w_{i+1,j} - w_{i+1,j+1} \quad (22)$$

for $i = 1, 2, \dots, b - 1$, and $j = 1, 2, \dots, b - 1$.

Proof: The proof is similar to the above, and is thus omitted.

From the above two properties, it can be concluded that only a small subclass of LI-filters and TD filters are equivalent. In this subclass, each filter possesses $2b - 1$ independent coefficients. Following the previous notation, denote the coefficients



Fig. 3. (Continued.) Experimental results. (q) and (r) are images of (o) and (p) filtered by a 3×3 linear filter that is configured from (a) and (o). (s) and (t) are images of (o) and (p) filtered by a 3×3 TD filter that is configured from (a) and (o).

on the i th level as \vec{w}_i , then

$$\vec{w}_i = \vec{w} + \alpha_i \vec{1} \tag{23}$$

where \vec{w} contains b independent coefficients. Since there are only $2b - 1$ independent coefficients, one of α_i can be set to zero. For convenience, we do not impose this condition. Substituting (23) into (6)

$$S(\vec{x}) = \sum_{i=1}^b (\vec{w} + \alpha_i \vec{1}) T_{x_i}(\vec{x}) \delta_i \tag{24}$$

$$= \vec{w} \sum_{i=1}^b T_{x_i}(\vec{x}) \delta_i + \sum_{i=1}^b (b + 1 - i) \alpha_i (x_{i_i} - x_{i_{i-1}}) \tag{25}$$

$$= \vec{w} \vec{x} + \vec{v} \vec{x}_s \tag{26}$$

where \vec{x}_s denotes the sorted \vec{x} . Hence, any filter in the subclass can be implemented as a linear filter and an order-statistic filter interconnected in parallel. For convenience, they will be referred to as LOS filters.

Equations (24)–(26) demonstrate the following two properties of LOS filters.

Property 4.3: An FIR is a LOS filter.

Property 4.4: An order statistic filter (L-filter) is a LOS filter.

The following property results from Property 4.4 immediately.

Property 4.5: An order statistic filter (L-filter) is a TD filter.

V. PERFORMANCE COMPARISON

According to (7), the output of TD filter $S(\vec{x})$ is a linear combination of $x_{i,j} \delta_i$ ($i = 1, 2, \dots, b$ and $j = 1, 2, \dots, b$). Just like linear filters, the optimal TD filter in this case under the least mean squares (LMS) criterion satisfies the following equation:

$$E[\vec{u} \vec{u}^t] \vec{w} = E[\vec{u} d] \tag{27}$$

where $\vec{u} = [x_{1,1} \delta_1, x_{1,2} \delta_1, \dots, x_{1,b} \delta_1, x_{2,1} \delta_2, \dots, x_{b,b} \delta_b]^t$, \vec{w} is the weight vector having the same dimension as \vec{u} , and d is the desired output.



Fig. 3. (*Continued.*) Experimental results. (u) and (v) are images of (o) and (p) filtered by a 3×3 LI-filter which is configured from (a) and (o). (w) and (x) are images of (o) and (p) filtered by a 3×3 LOS filter which is configured from (a) and (o). (y) and (z) are images of (o) and (p) filtered by a 3×3 median filter.

In practice, the expectation is replaced by the averaging operator, and (27) is simplified as

$$\mathbf{R}\vec{w} = \vec{P} \tag{28}$$

where $\mathbf{R} = \overline{(\vec{u}\vec{u}^t)}$ and $\vec{P} = \overline{(\vec{u}\vec{d})}$, and $\overline{(\cdot)}$ is the averaging operator.

When matrix \mathbf{R} is nonsingular, $\vec{w} = \mathbf{R}^{-1}\vec{P}$. Otherwise, the number of solutions will be infinite. Any solution can be

TABLE I
MAE AND RMSE OF NOISY (GAUSSIAN NOISE)
IMAGES AND OUTPUT OF VARIOUS FILTERS

	lena		woman1	
	MAE	RMSE	MAE	RMSE
Noisy image	21.90	27.32	22.37	28.04
3 × 3 Linear filtering	9.47	12.34	10.59	14.17
3 × 3 TD filtering	9.42	12.24	10.57	14.15
3 × 3 LI filtering	9.38	12.18	10.50	14.01
3 × 3 LOS filtering	9.44	12.28	10.57	14.14
3 × 3 L-filtering	9.57	12.68	10.84	14.87
3 × 3 Median filtering	10.95	14.26	12.03	16.32

TABLE II
MAE AND RMSE OF NOISY (SALT-AND-PEPPER
NOISE) IMAGES AND OUTPUT OF VARIOUS FILTERS

	lena		woman1	
	MAE	RMSE	MAE	RMSE
Noisy image	20.43	56.37	20.50	54.06
3 × 3 Linear filtering	16.94	22.60	17.64	23.29
3 × 3 TD filtering	4.05	9.08	5.67	11.95
3 × 3 LI filtering	4.41	9.24	5.64	11.95
3 × 3 LOS filtering	4.67	9.64	6.39	12.71
3 × 3 L-filtering	4.61	9.68	6.38	12.81
3 × 3 Median filtering	4.35	9.87	6.17	13.10

expressed as

$$\vec{w} = \vec{w}_P + \vec{w}_N \quad (29)$$

where \vec{w}_P satisfies (28), and \vec{w}_N belongs to the null space, i.e.,

$$\mathbf{R}\vec{w}_N = 0. \quad (30)$$

Therefore, each solution yields the same mean square error (MSE):

$$\text{MSE} = -\vec{w}_P^t \mathbf{R} \vec{w}_P + \bar{d}^2. \quad (31)$$

In our experiment, two types of noise are used: Gaussian noise and salt-and-pepper noise. To configure a filter, the original lena image is referenced as the desired output, and its noisy version is employed as the input. That is, given the two images, the weights of the optimal TD filter is obtained by solving (28). This filter is then used to filter the noisy Lena image and the woman1 image corrupted by the same type of noise.

By the same token, the optimal LI-filters, LOS filters, and linear filters under the LMS criterion also satisfy (28), except that the definition of \vec{u} and the size of the vectors are different. In order to configure LI-filters, \vec{u} is defined as $\vec{u} = [x_{l_1}\delta(1-l_1), x_{l_1}\delta(2-l_1), \dots, x_{l_1}\delta(b-l_1), x_{l_2}\delta(1-l_2), \dots, x_{l_b}\delta(b-l_b)]^t$, where $\delta(n)$ is one if $n = 0$, and zero, otherwise; $\vec{u} = [x_1, x_2, \dots, x_b, x_{l_2} - x_{l_1}, \dots, x_{l_b} - x_{l_{b-1}}]^t$ for LOS filters, $\vec{u} = [x_{l_1}, x_{l_2}, \dots, x_{l_b}]^t$ for L filters, and $\vec{u} = [x_1, x_2, \dots, x_b]^t$ for linear filters.

Fig. 3 shows the original images, noisy images, and filtered images of lena and woman1, respectively. Mean absolute error (MAE) and root mean square error (RMSE) of the noisy images and filtered images are tabulated in Tables I and II. For comparison purposes, results of median filtering are also

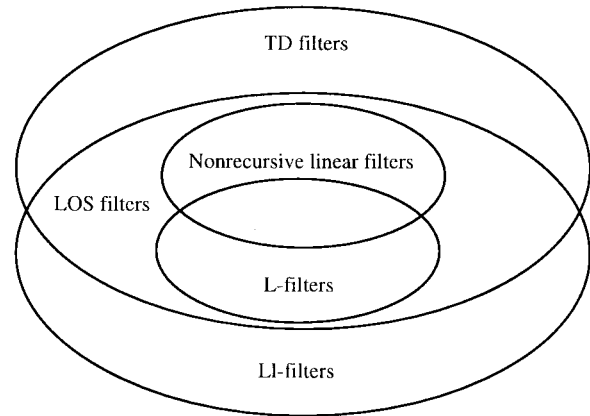


Fig. 4. Relationship among different classes of filters.

included in the figures and tables. The resolution of all images is 256×256 , 8 b/pixel. Variances of the original Lena and woman1 images are 2734 and 1811, respectively. The window size used in our experiments is 3×3 .

VI. CONCLUSIONS

The relationship among several types of filters is illustrated in Fig. 4. The intersection between the TD filters and LI-filters forms the LOS filters, a simple addition of linear operation and L-filtering. It is evident that LOS filters generalize FIR and L-filters, and thus FIR and L-filters are subsets of TD filters and LI-filters.

According to Tables I and II, the performance in suppressing Gaussian noise among TD filters, LI-filters, LOS filters and linear filters are similar. Median filtering performs poorly in suppressing Gaussian noise as expected. Among the filters tested on these images, LI achieves the best performance in suppressing Gaussian noise while TD suppresses salt-and-pepper noise the best.

With a window size of b , it requires b coefficients to configure a linear filter, b^2 coefficients to configure a TD or LI-filter, and only $2b - 1$ coefficients to configure a LOS filter. Therefore, LOS filters provide a trade-off between performance and complexity.

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