Physics 111: Mechanics
Lecture 10

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Phys. 111 (Part I): Translational Mechanics

- Motion of point bodies
- Translational motion. Size and shape not considered
- dynamics $\sum F_{ext} = ma$
- conservation laws: energy & momentum

Phys. 111 (Part II): Rotational Mechanics

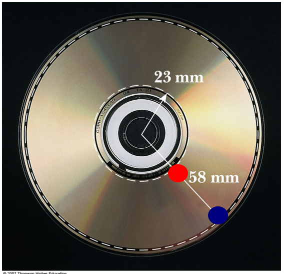
- motion of "Rigid Bodies" (extended, finite size)
- rotation + translation, more complex motions possible
- rigid bodies: fixed size & shape, orientation matters
- dynamics

$$\sum F_{ext} = ma_{cm} \qquad \sum \tau_z = I\alpha_z$$

- rotational modifications to energy conservation
- conservation laws: energy & angular momentum

Chapter 9 Rotation of Rigid Bodies

- 9.1 Angular Velocity and Acceleration
- 9.2 Rotation with Constant Angular Acceleration
- 9.3 Relating Linear and Angular Kinematics
- 9.4 Energy in Rotational Motion
- 9.5 Parallel-Axis Theorem
- Moments-of-Inertia Calculations



Rigid Object

- A rigid object is one that is nondeformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible
- This simplification allows analysis of the motion of an extended object

Angle and Radian

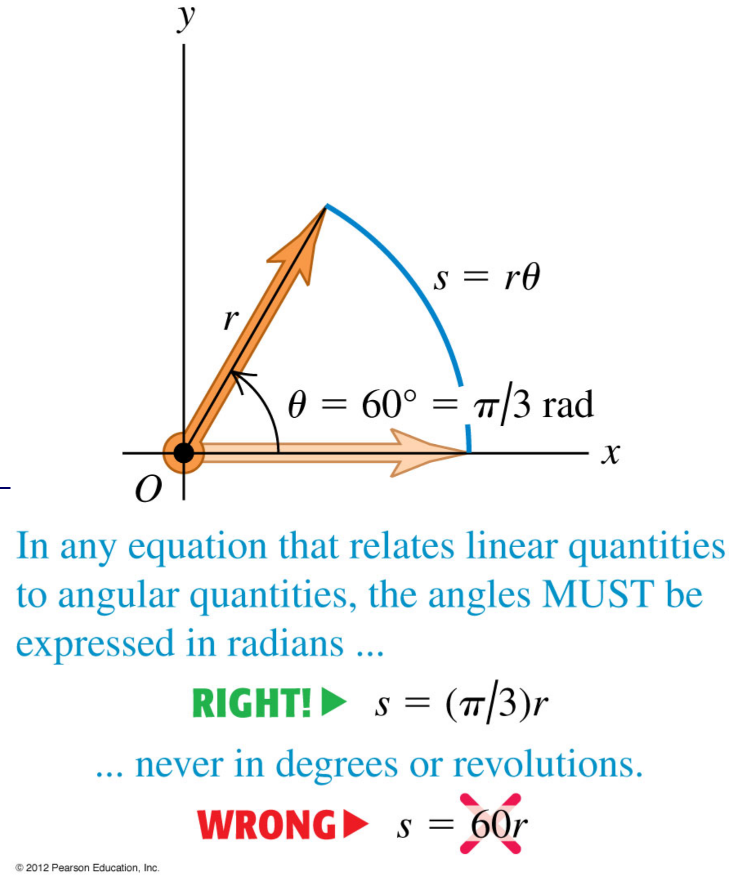
- What is the circumference S?

$$s = (2\pi)r \qquad 2\pi = \frac{s}{r}$$

- θ can be defined as the arc length s along a circle divided by the radius r:

$$\theta = \frac{s}{r}$$

- θ is a pure number, but commonly is given the artificial unit, radian ("rad")



- Whenever using rotational equations, you **MUST** use angles expressed in radians

Conversions

- Comparing degrees and radians

$$2\pi(rad) = 360^\circ \qquad \pi(rad) = 180^\circ$$

- Converting from degrees to radians

$$\theta\,(rad) = \frac{\pi}{180^\circ}\,\theta\,(degrees)$$

- Converting from radians to degrees

$$\theta(\deg rees) = \frac{180^\circ}{\pi}\theta(rad) \qquad 1\,rad = \frac{360^\circ}{2\pi} = 57.3^\circ$$

- Converting from revolutions to radians

1 revolution = 2π (rad) = 360° rpm: revolutions per minute

Conversion

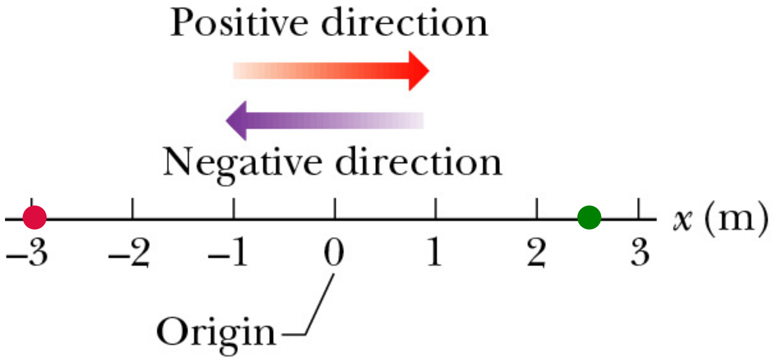
- A waterwheel turns at 360 revolutions per hour. Express this figure in radians per second.

A) 3.14 rad/s
B) 6.28 rad/s
C) 0.314 rad/s
D) 0.628 rad/s

$$1\frac{revolution}{s} = 2\pi\, rad/s$$

One Dimensional Position x

- What is motion? Change of position over time.
- How can we represent position along a straight line?
- Position definition:
  - Defines a starting point: origin (x = 0), x relative to origin
  - Direction: positive (right or up), negative (left or down)
  - It depends on time: t = 0 (start clock), x(t=0) does not have to be zero.
- Position has units of [Length]: meters.

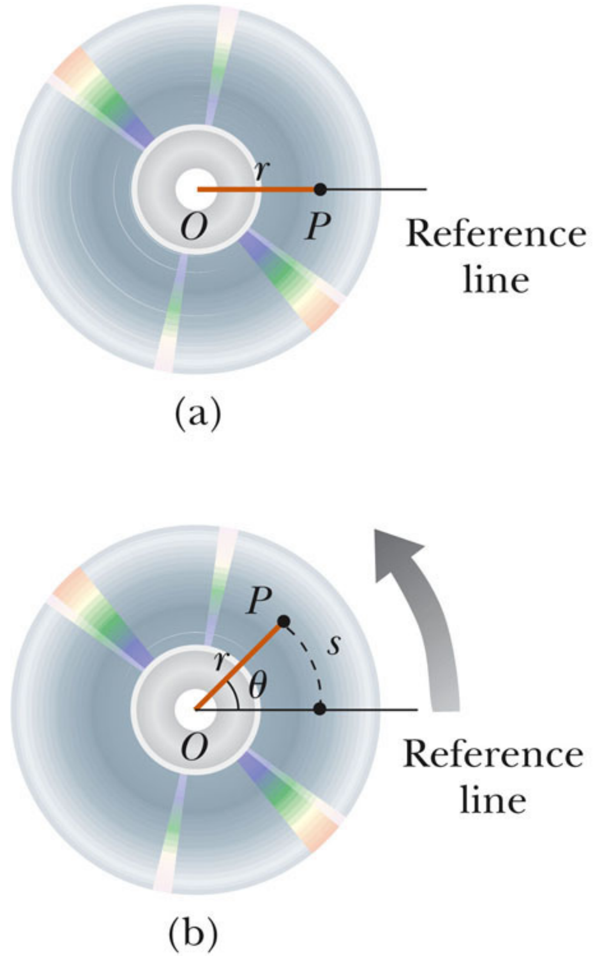


$x = +2.5$ m

$x = -3$ m

Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin
- As the particle moves, the only coordinate that changes is θ
- As the particle moves through θ, it moves though an arc length s.
- The angle θ, measured in radians, is called the angular position.



Displacement

- Displacement is a change of position in time.
- Displacement: $\Delta x = x_f(t_f) - x_i(t_i)$
  - *f* stands for final and *i* stands for initial.
- It is a vector quantity.
- It has both magnitude and direction: + or - sign
- It has units of [length]: meters.

$x_1(t_1) = +2.5\ m$

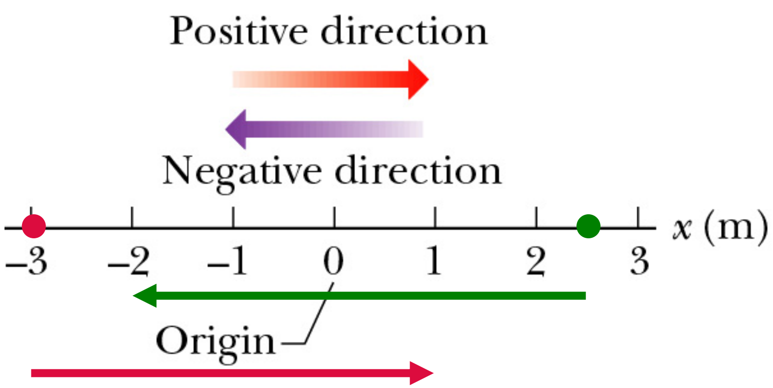
$x_2(t_2) = -2.0\ m$

$\Delta x = -2.0\ m - 2.5\ m = -4.5\ m$

$x_1(t_1) = -3.0\ m$

$x_2(t_2) = +1.0\ m$

$\Delta x = +1.0\ m + 3.0\ m = +4.0\ m$

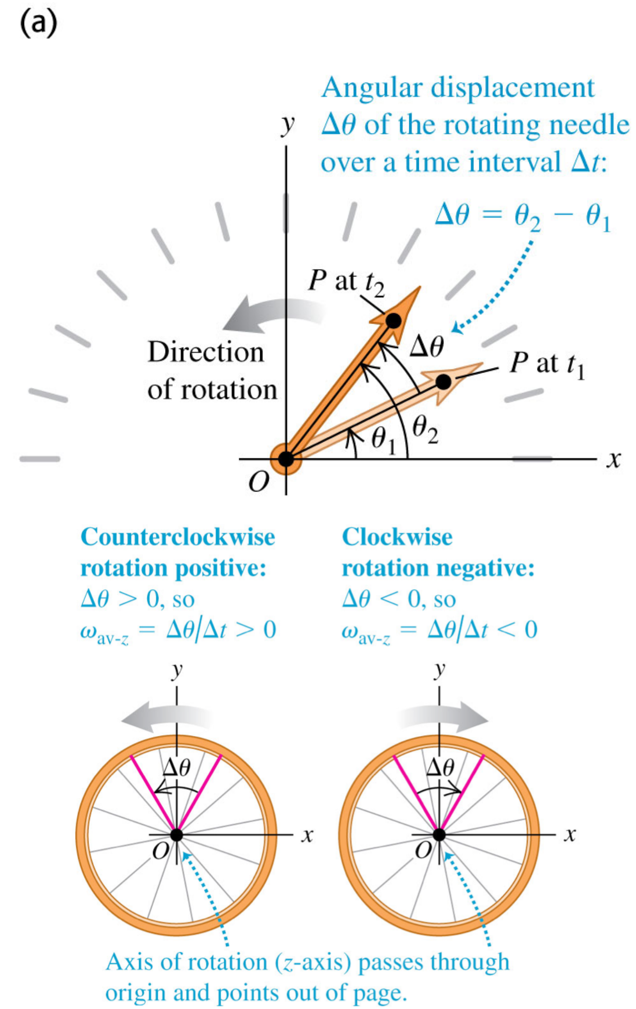


Angular Displacement

- The angular displacement is defined as the angle the object rotates through during some time interval

$$\Delta\theta = \theta_f - \theta_i$$

- SI unit: radian (rad)
- A counterclockwise rotation is positive.
- A clockwise rotation is negative.



Velocity

- Velocity is the rate of change of position
- Average velocity

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{\Delta t}$$

displacement

- Average speed

$$S_{avg} = total\ distance/total\ time$$

distance

- Instantaneous velocity

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x_f - x_i}{\Delta t}$$

displacement

Average and Instantaneous Angular Velocity

- The *average* angular velocity, ω_{avg}, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

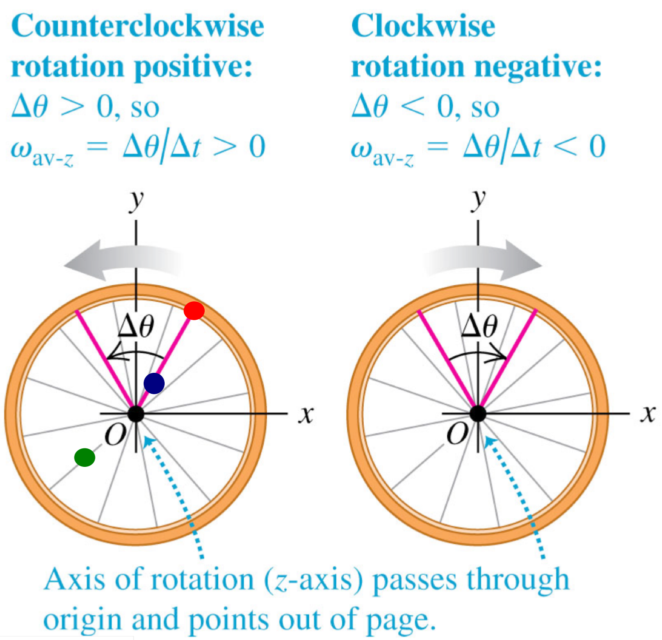
- The *instantaneous* angular velocity is defined as the limit of the average velocity as the time interval approaches zero

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

- SI unit: radian per second (rad/s)

Angular Velocity: + or - ?

- Angular velocity positive if rotating in counterclockwise
- Angular velocity will be negative if rotating in clockwise
- Every point on the rotating rigid object has the same angular velocity



Average Acceleration

- Changing velocity (non-uniform) means an acceleration is present.
- Acceleration is the rate of change of velocity.
- Acceleration is a vector quantity.
- Acceleration has both magnitude and direction.
- Acceleration has a unit of [length/time^2]: m/s^2.
- Definition:
  - Average acceleration $$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$
  - Instantaneous acceleration $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt}$$

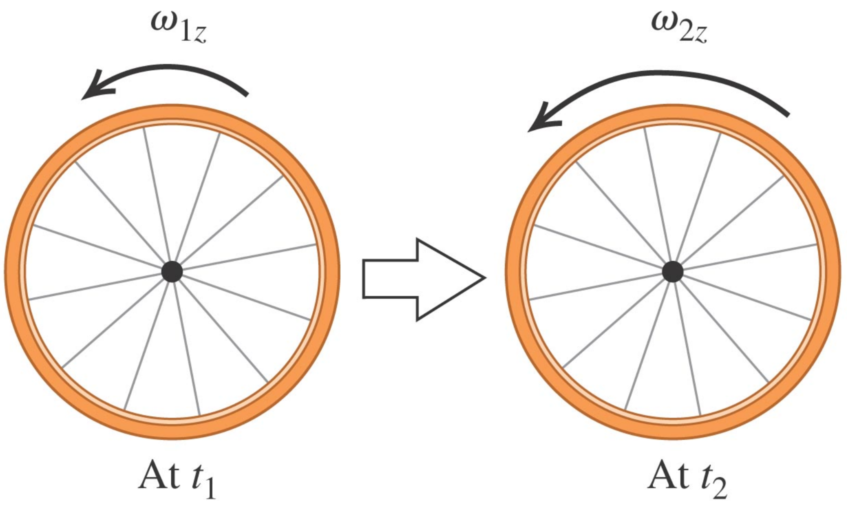
Average Angular Acceleration

- The average angular acceleration, α, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

The average angular acceleration is the change in angular velocity divided by the time interval:

$$\alpha_{\text{av-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$



Instantaneous Angular Acceleration

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

- SI Units of angular acceleration: rad/s²
- Positive angular acceleration is in the counterclockwise direction.
  - if an object rotating counterclockwise is speeding up
  - if an object rotating clockwise is slowing down
- Negative angular acceleration is in the clockwise direction.
  - if an object rotating counterclockwise is slowing down
  - if an object rotating clockwise is speeding up

Rotational Kinematics

- A number of parallels exist between the equations for rotational motion and those for linear motion.

$$v_{avg} = \frac{x_f - x_i}{t_f - t_i} = \frac{\Delta x}{\Delta t} \qquad \omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}$$

- Under **constant angular acceleration**, we can describe the motion of the rigid object using a set of kinematic equations
  - These are similar to the kinematic equations for linear motion
  - The rotational equations have the same mathematical form as the linear equations

Comparison Between Rotational and Linear Equations

Table 9.1 Comparison of Linear and Angular Motion with Constant Acceleration

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x (x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z (\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_x + v_{0x})t$	$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$

Angular Motion

- At t = 0, a wheel rotating about a fixed axis at a constant angular acceleration has an angular velocity of 2.0 rad/s. Two seconds later it has turned through 5.0 complete revolutions. Find the angular acceleration of this wheel?

  A. 17 rad/s²
  B. 14 rad/s²
  C. 20 rad/s²
  D. 23 rad/s²
  E. 12 rad/s²

$$\alpha_z = \text{constant}$$

$$\omega_z = \omega_{0z} + \alpha_z t$$

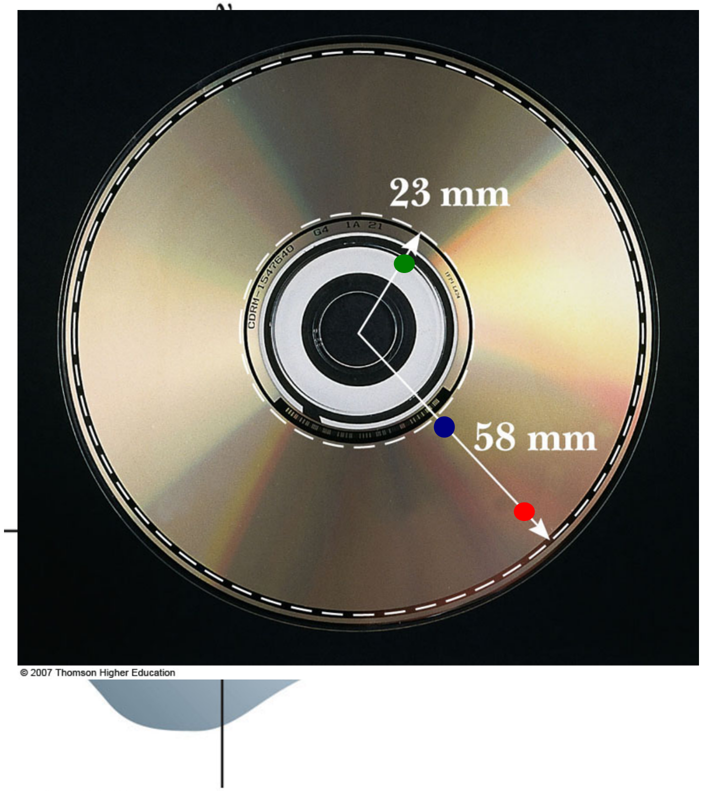
$$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2$$

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$$

$$\theta - \theta_0 = \frac{1}{2}(\omega_z + \omega_{0z})t$$

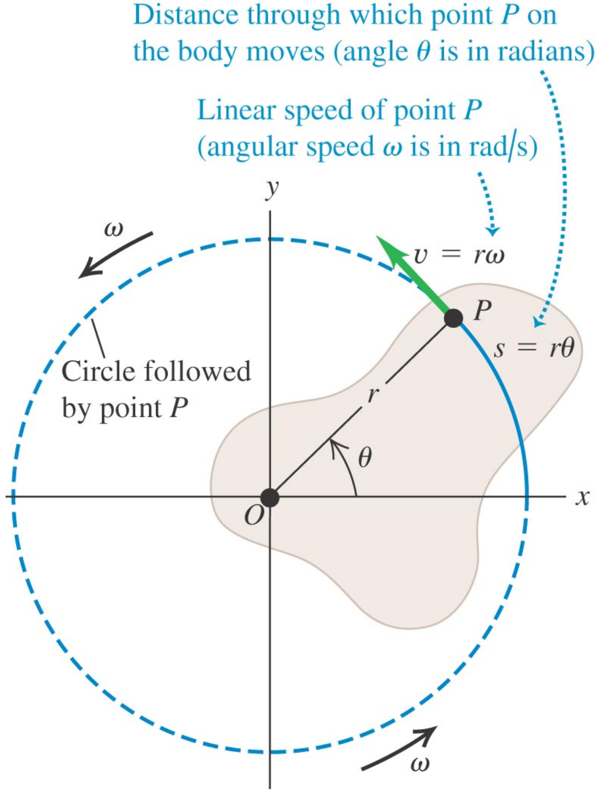
Relating Angular and Linear Kinematics

- Every point on the rotating object has the **same angular motion** (**angular displacement**, **angular velocity, angular acceleration**)
- Every point on the rotating object does **not** have the same linear motion
- Displacement $s = \theta r$
- Velocity $v = \omega r$
- Acceleration $a = \alpha r$



Velocity Comparison

- The linear velocity is always tangent to the circular path
  - Called the tangential velocity
- The magnitude is defined by the tangential velocity



$$\Delta\theta = \frac{\Delta s}{r}$$

$$\frac{\Delta\theta}{\Delta t} = \frac{\Delta s}{r\Delta t} = \frac{1}{r}\frac{\Delta s}{\Delta t}$$

$$\omega = \frac{v}{r} \quad or \quad v = r\omega$$

Acceleration Comparison

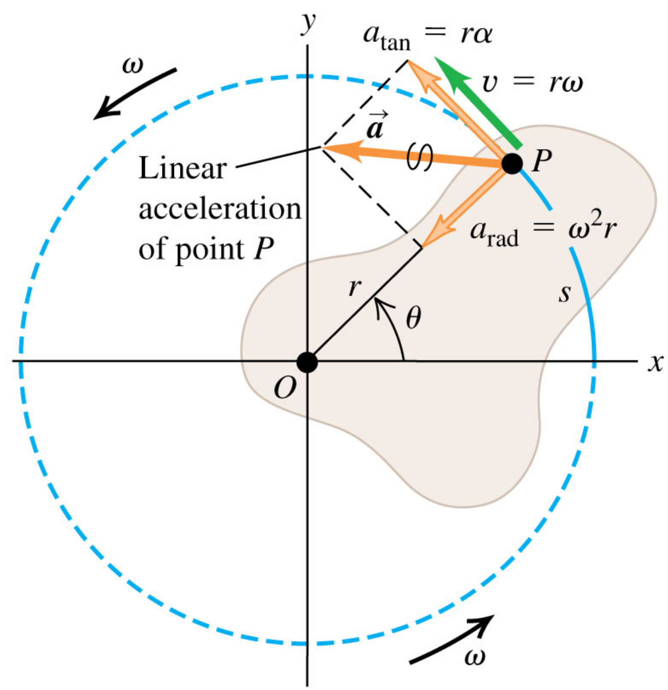
- The tangential acceleration is the derivative of the tangential velocity

$$\Delta v = r\Delta\omega$$

$$\frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t} = r\alpha$$

$$a_t = r\alpha$$

Radial and tangential acceleration components:
- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).



Velocity and Acceleration Note

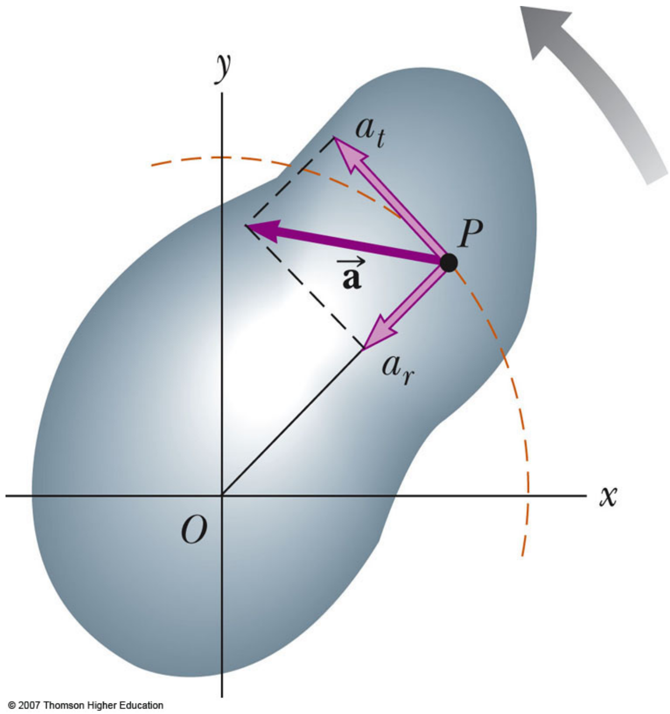
- All points on the rigid object will have the same angular speed, but not the same tangential speed
- All points on the rigid object will have the same angular acceleration, but not the same tangential acceleration
- The tangential quantities depend on r, and r is not the same for all points on the object

$$\omega = \frac{v}{r} \quad or \quad v = r\omega \qquad a_t = r\alpha$$

Centripetal Acceleration

- An object traveling in a circle, even though it moves with a constant speed, will have an acceleration
  - Therefore, each point on a rotating rigid object will experience a centripetal acceleration

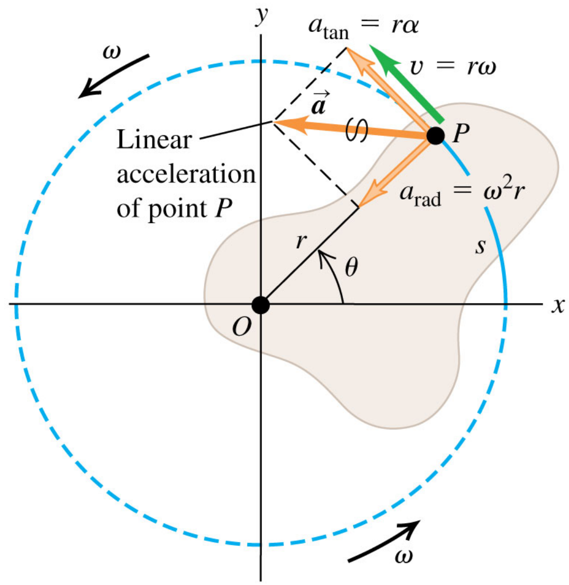
$$a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$



Resultant Acceleration

- The tangential component of the acceleration is due to changing speed
- The centripetal component of the acceleration is due to changing direction
- Total acceleration can be found from these components

Radial and tangential acceleration components:
- $a_{rad} = \omega^2 r$ is point P's centripetal acceleration.
- $a_{tan} = r\alpha$ means that P's rotation is speeding up (the body has angular acceleration).

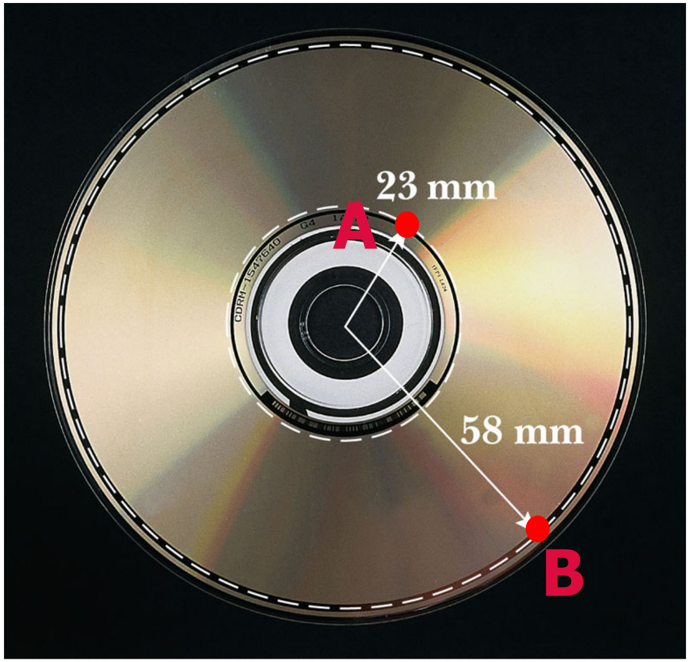


$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4}$$

Angular and Linear Quantities

- For a rigid rotational CD, which statement below is true for the two points A and B on this CD?

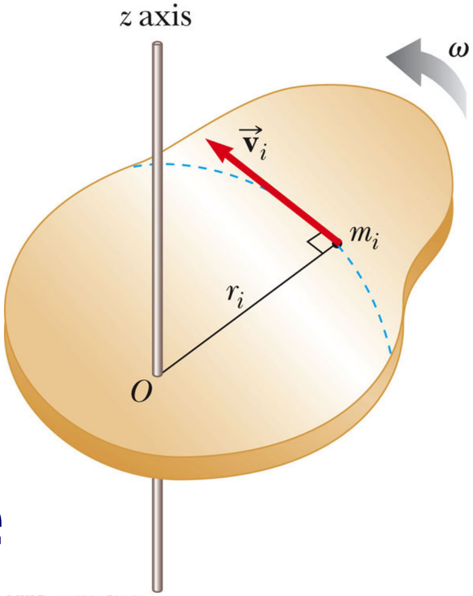
A) Same distance travelled in 1s
B) Same linear velocity
C) Same centripetal acceleration
D) Same linear acceleration
E) Same angular velocity



Rotational Kinetic Energy

- An object rotating about z axis with an angular speed, ω, has rotational kinetic energy
- Each particle has a kinetic energy of
  - $K_i = ½\ m_i v_i^2$
- Since the tangential velocity depends on the distance, r, from the axis of rotation, we can substitute

$v_i = \omega r_i$

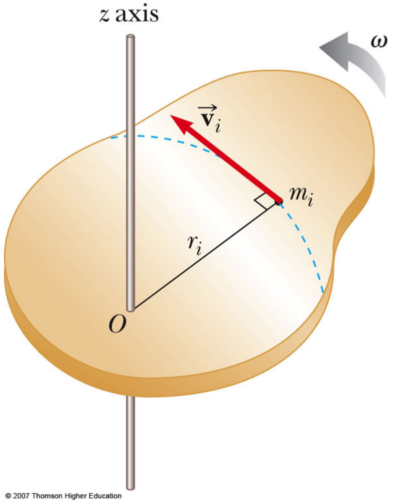


Rotational Kinetic Energy, cont

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$



- Where I is called the moment of inertia

Rotational Kinetic Energy, final

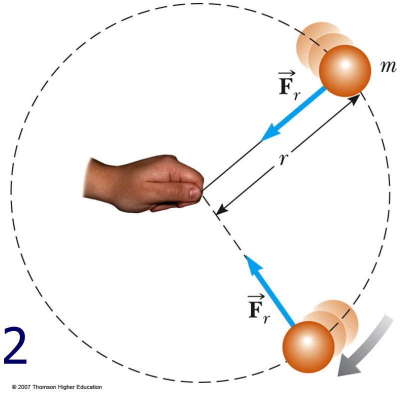
- There is an analogy between the kinetic energies associated with linear motion ($K = \frac{1}{2} mv^2$) and the kinetic energy associated with rotational motion ($K_R = \frac{1}{2} I\omega^2$)
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object
- Units of rotational kinetic energy are Joules (J)

Moment of Inertia of Point Mass

- For a single particle, the definition of moment of inertia is

$$I = mr^2$$

  - m is the mass of the single particle
  - r is the rotational radius

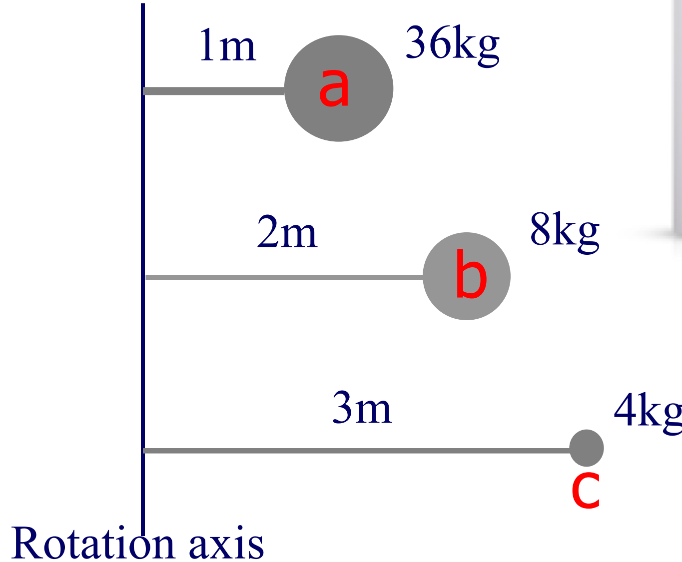


- SI units of moment of inertia are kg·m²
- Moment of inertia and mass of an object are different quantities
- It depends on both the quantity of matter and its distribution (through the r^2 term)

Moment of Inertia

- The figure shows three small spheres that rotate about a vertical axis. The perpendicular distance between the axis and the center of each sphere is given. Rank the three spheres according to their moment of inertia about that axis, greatest first ?

$$I = mr^2$$



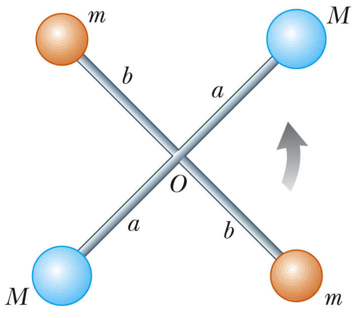
A) a, b, c
B) b, a, c
C) c, b, a
D) all tie
E) a and c tie, b

Moment of Inertia of Point Mass

- For a composite particle, the definition of moment of inertia is

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots$$

  - m_i is the mass of the ith single particle
  - r_i is the rotational radius of ith particle



(b)

- SI units of moment of inertia are kg·m^2
- Consider an unusual baton made up of four sphere fastened to the ends of very light rods
- Find *I* about an axis perpendicular to the page and passing through the point *O* where the rods cross

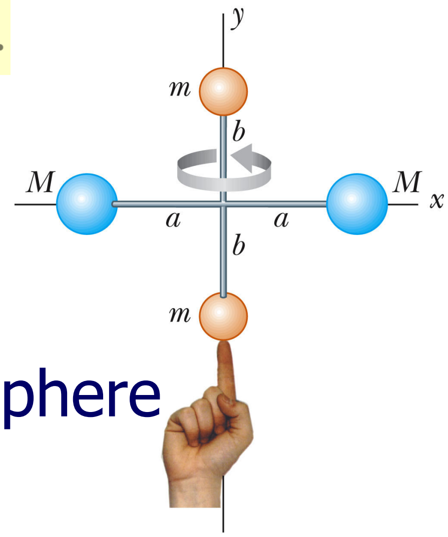
$$I = \sum m_i r_i^2 = mb^2 + Ma^2 + mb^2 + Ma^2 = 2Ma^2 + 2mb^2$$

Moment of Inertia of Point Mass

- For a composite particle, the definition of moment of inertia is

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + m_4 r_4^2 + \ldots$$

  - m_i is the mass of the ith single particle
  - r_i is the rotational radius of ith particle
- SI units of moment of inertia are kg·m^2
- Consider an unusual baton made up of four sphere fastened to the ends of very light rods
- Find I about axis y



(a)

$$I = \sum m_i r_i^2 = Mr_1^2 + Mr_2^2 + mr_3^2 + mr_4^2 = Ma^2 + Ma^2 + 0 + 0$$

Moment of Inertia of Extended Objects

- Divided the extended objects into many small volume elements, each of mass Δm_i
- We can rewrite the expression for I in terms of Δm

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

- With the small volume segment assumption,

$$I = \int \rho r^2 dV$$

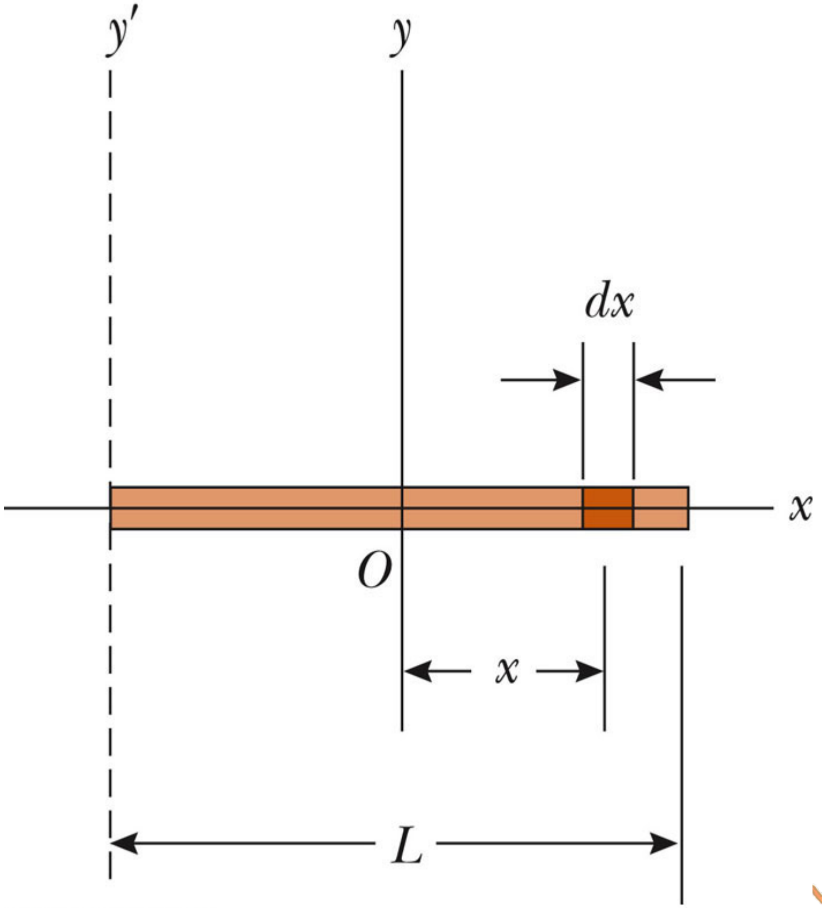
- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known

Moment of Inertia of a Uniform Rigid Rod

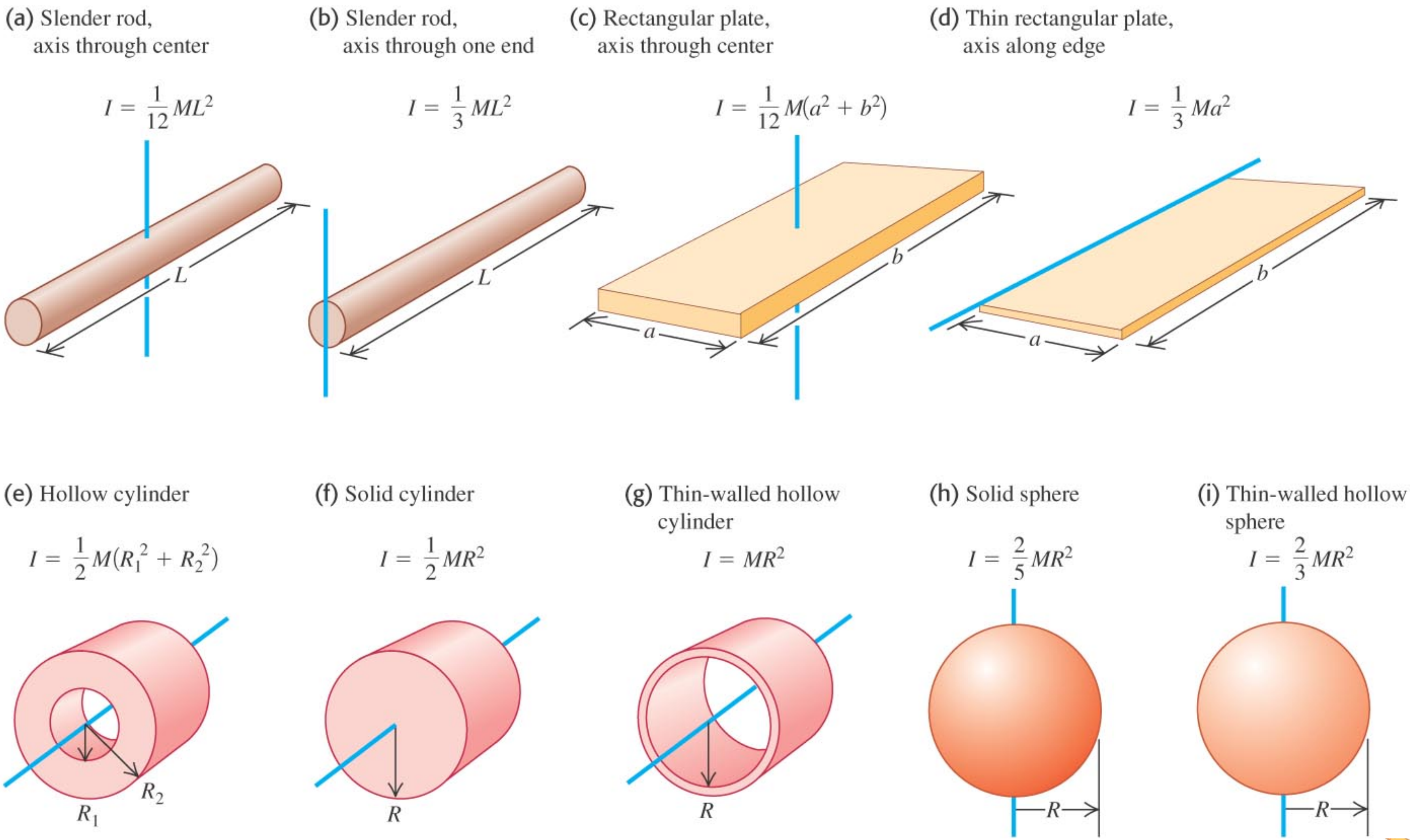
- The shaded area has a mass
  - $dm = \lambda\, dx \quad \lambda = M/L$
- Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{12} ML^2$$



M-I for some other common shapes

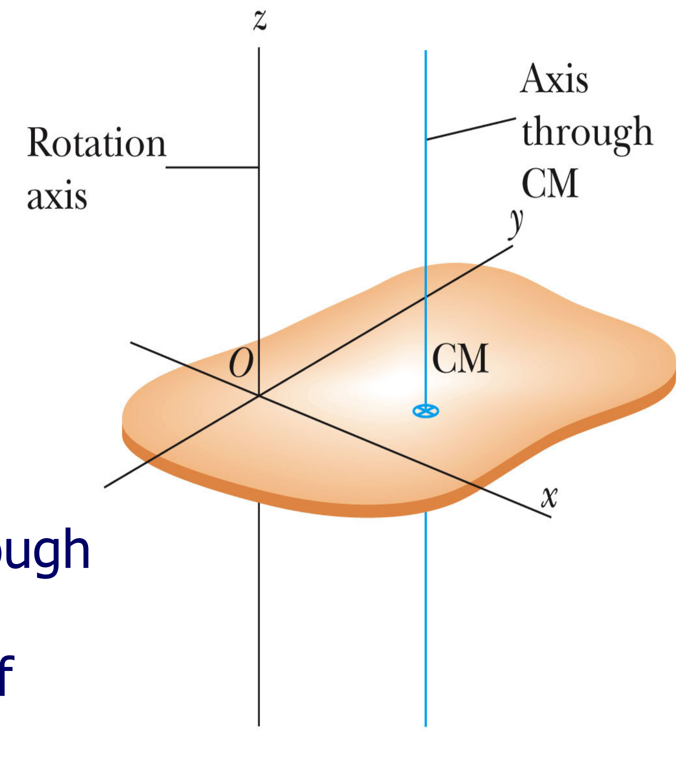


Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations
- The theorem states

$$I = I_{CM} + MD^2$$

  - I is about any axis parallel to the axis through the center of mass of the object
  - I_{CM} is about the axis through the center of mass
  - D is the distance from the center of mass axis to the arbitrary axis



(b)

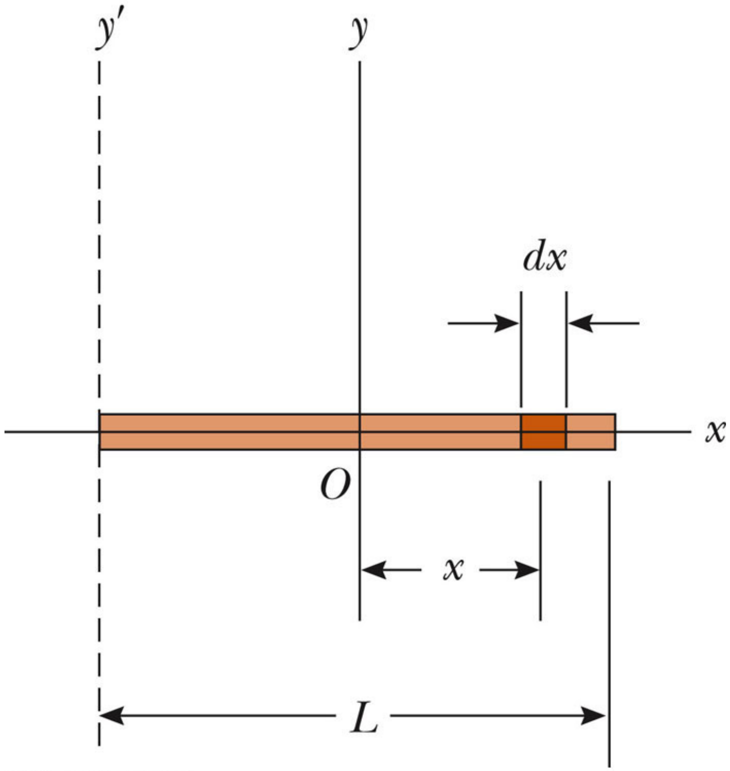
Moment of Inertia of a Uniform Rigid Rod

- The moment of inertia about y is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{12} ML^2$$

- The moment of inertia about y' is



$$I_{y'} = I_{CM} + MD^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$$

Chap. 9 Summary

Rotational Kinematics

$$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (9.3)$$

$$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \qquad (9.5), (9.6)$$

$$\theta = \theta_0 + \omega_{0z}t + \tfrac{1}{2}\alpha_z t^2 \qquad (9.11)$$

(constant α_z only)

$$\theta - \theta_0 = \tfrac{1}{2}(\omega_{0z} + \omega_z)t \qquad (9.10)$$

(constant α_z only)

$$\omega_z = \omega_{0z} + \alpha_z t \qquad (9.7)$$

(constant α_z only)

$$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0) \qquad (9.12)$$

(constant α_z only)

Relating linear and angular kinematics

$$v = r\omega \qquad (9.13)$$

$$a_{\tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \qquad (9.14)$$

$$a_{\rm rad} = \frac{v^2}{r} = \omega^2 r \qquad (9.15)$$

Moment of inertia and rotational kinetic energy

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots = \sum_i m_i r_i^2 \qquad (9.16)$$

$$K = \tfrac{1}{2} I\omega^2 \qquad (9.17)$$

The parallel-axis theorem

$$I_P = I_{\rm cm} + Md^2 \qquad (9.19)$$