Physics 111: Mechanics
Lecture 11

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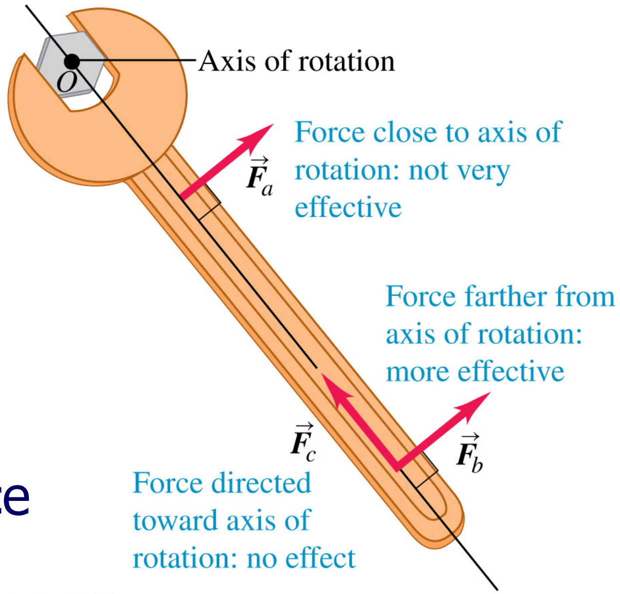
Textbook Chapter 10: Dynamics of Rotational Motion

- 10.1 Torque
- 10.2 Torque and Angular Acceleration for a Rigid Body
- 10.3 Rigid-Body Rotation about a Moving Axis
- 10.4 Work and Power in Rotational Motion (partially covered in previous lecture)
- 10.5 Angular Momentum
- 10.6 Conservation of Angular Momentum
- 10.7* Gyroscopes and Precession

Dynamics of Rotation

Force vs. Torque

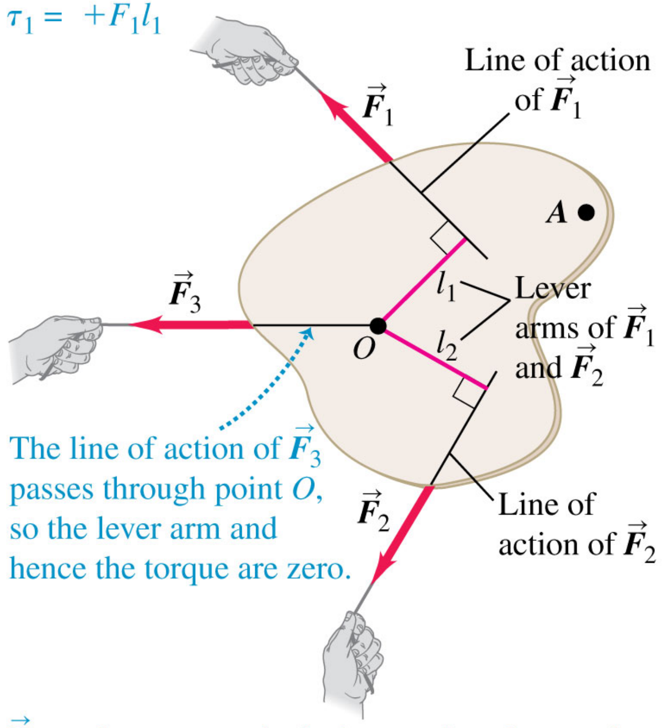
- Forces cause accelerations
- What cause angular accelerations?
- A door is free to rotate about an axis through O
- There are three factors that determine the effectiveness of the force in loosening the tight bolt:
  - The *magnitude* of the force
  - The *position* of the application of the force
  - The *angle* at which the force is applied



Torque Definition

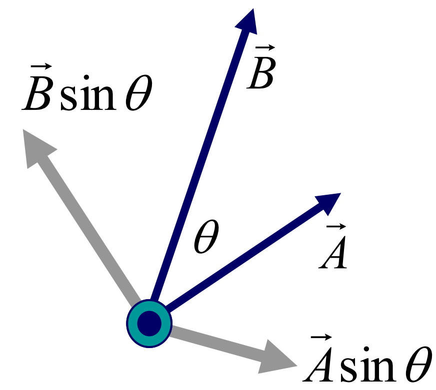
- Torque, τ, is the tendency of a force to rotate an object about some axis
- Let **F** be a force acting on an object, and let **r** be a position vector from a rotational center to the point of application of the force, with **F** perpendicular to **r**. The magnitude of the torque is given by

$$\tau = rF$$

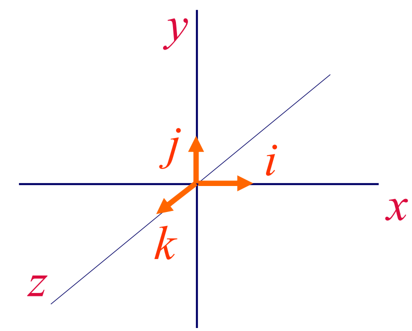


Cross Product

$$\vec{C} = \vec{A} \times \vec{B}$$

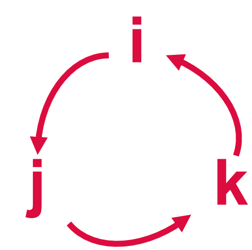


- The cross product of two vectors says something about how perpendicular they are.
- Magnitude: $\left|\vec{C}\right| = \left|\vec{A} \times \vec{B}\right| = AB\sin\theta$
  - θ is smaller angle between the vectors
  - Cross product of any parallel vectors = zero
  - Cross product is maximum for perpendicular vectors
  - Cross products of Cartesian unit vectors:



$$\hat{i} \times \hat{j} = \hat{k};\ \hat{i} \times \hat{k} = -\hat{j};\ \hat{j} \times \hat{k} = \hat{i}$$

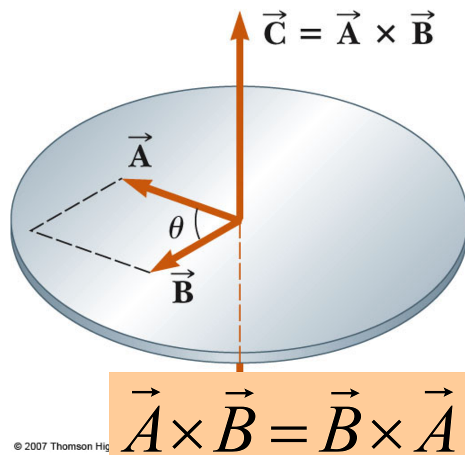
$$\hat{i} \times \hat{i} = 0;\ \hat{j} \times \hat{j} = 0;\ \hat{k} \times \hat{k} = 0$$



Cross Product

- Direction: C perpendicular to both A and B (right-hand rule)
  - Place A and B tail to tail
  - Right hand, not left hand
  - Four fingers are pointed along the first vector A
  - "sweep" from first vector A into second vector B through the smaller angle between them
  - Your outstretched thumb points the direction of C
- First practice

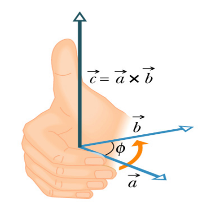
$$\vec{A}\times\vec{B}=\vec{B}\times\vec{A}?$$



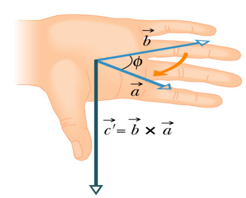
$$\vec{A}\times\vec{B}=\vec{B}\times\vec{A}?$$

Right-hand rule

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$



(a)



(b)

Torque Units and Direction

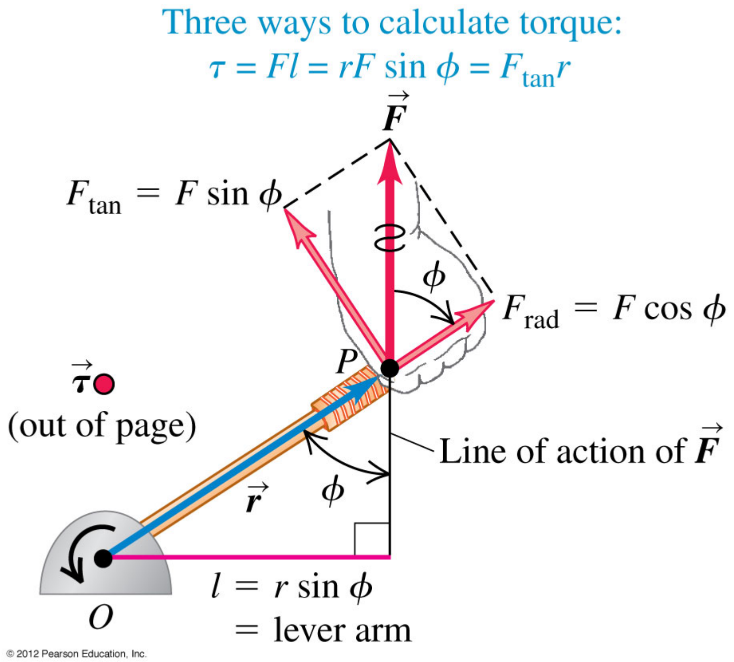
- The SI units of torque are N·m
- Torque is a vector quantity

$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Torque magnitude is given by

$$\tau = rF\sin\phi = Fl$$

- Torque will have direction
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative

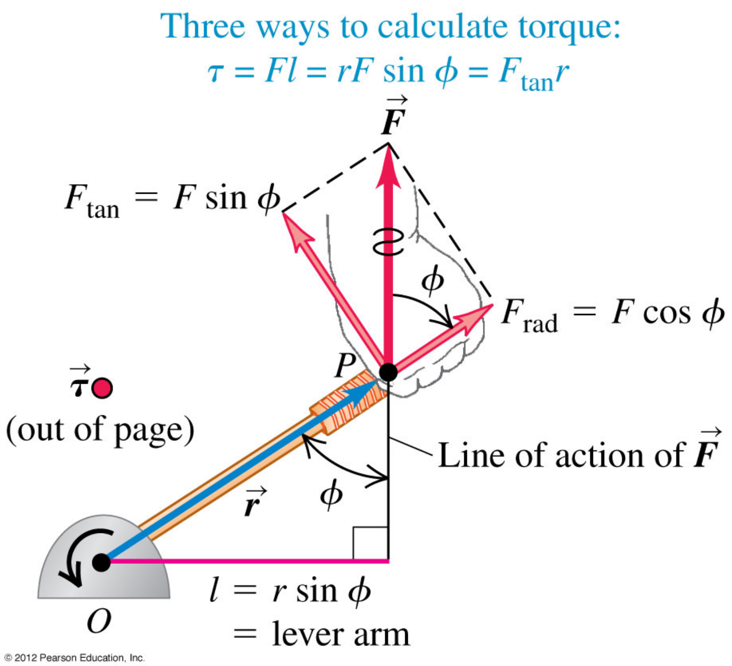


Understand sinϕ

- The component of the force ($F\cos\phi$) has no tendency to produce a rotation
- The component of the force ($F\sin\phi$) causes it to rotate
- The **moment arm**, *l*, is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force

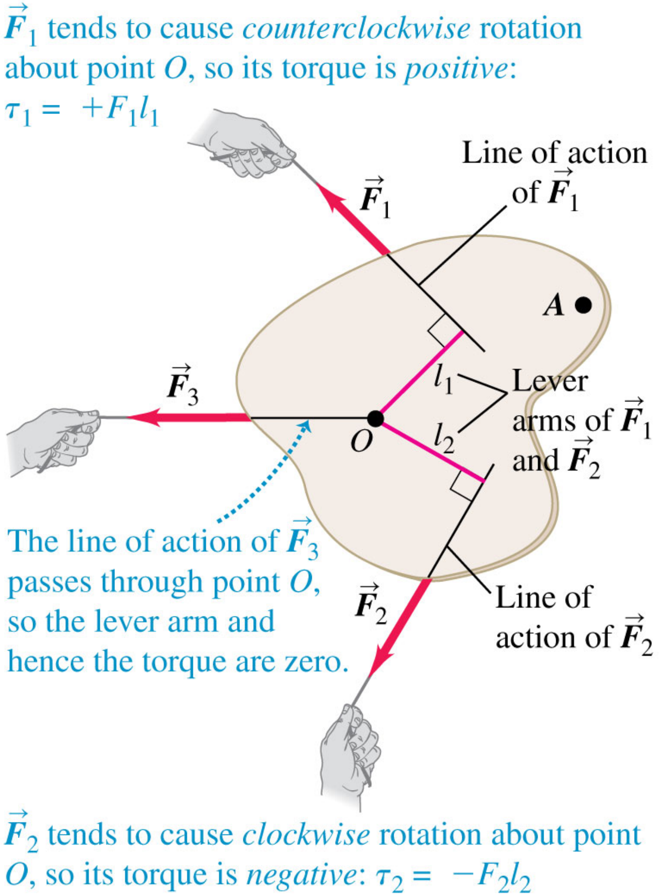
$$l = r\sin\phi$$

$$\tau = rF\sin\phi = Fl$$



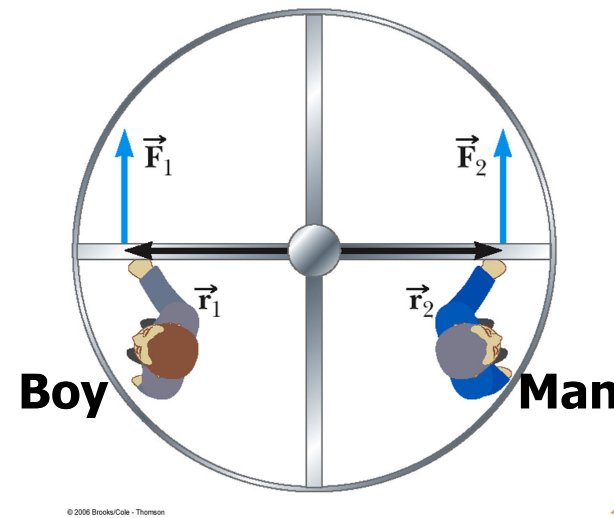
Net Torque

- The force $\vec{F}_1$ will tend to cause a counterclockwise rotation about *O*
- The force $\vec{F}_2$ will tend to cause a clockwise rotation about *O*
- $\Sigma\tau = \tau_1 + \tau_2 + \tau_3 = F_1 l_1 - F_2 l_2$
- If $\Sigma\tau \neq 0$, starts rotating
- If $\Sigma\tau = 0$, rotation rate does not change



Battle of the Revolving Door

- A man and a boy are trying to use a revolving door. The man enters the door on the right, pushing with 200 N of force directed perpendicular to the door and 0.60 m from the hub, while the boy exerts a force of 100 N perpendicular to the door, 1.25 m to the left of the hub. Finally, the door will



A) Rotate in counterclockwise

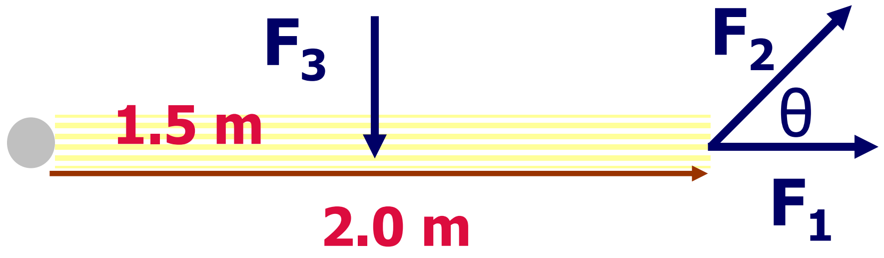
B) Rotate in clockwise

C) Stay at rest

D) Not enough information is given

The Swinging Door

- Two forces F_1 and F_2 are applied to the door, as shown in figure. Suppose a wedge is placed 1.5 m from the hinges on the other side of the door. What minimum force F_3 must the wedge exert so that the force applied won't open the door? Assume $F_1 = 150$ N, $F_2 = 300$ N, $\theta = 30^\circ$



$$\tau = \tau_1 + \tau_2 + \tau_3 = 0$$

$$\tau_1 = F_1 r_1 \sin\theta_1 = 150 \times 2 \sin 0^\circ = 0$$

$$\tau_2 = F_2 r_2 \sin\theta_2 = 300 \times 2 \sin 30^\circ = 300\text{Nm}$$

$$\tau_3 = -F_3 r_3 \sin\theta_3 = -1.5 F_3$$

$$0 + 300 - 1.5 F_3 = 0 \qquad F_3 = 200\ \text{Nm}$$

Newton's Second Law for a Rotating Object

- When a rigid object is subject to a net torque (≠0), it undergoes an angular acceleration

$$\Sigma\tau = I\alpha$$

- The angular acceleration is directly proportional to the net torque
- The angular acceleration is inversely proportional to the moment of inertia of the object
- The relationship is analogous to

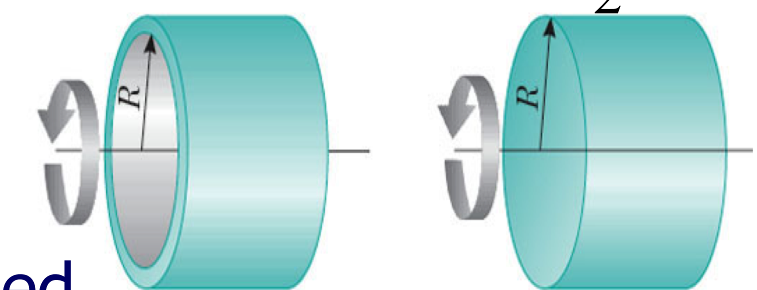
$$\sum F = ma$$

Newton 2nd Law in Rotation

- The two rigid objects shown in figure have the same mass, radius, and initial angular speed. If the same braking torque is applied to each, which takes longer to stop?

$\Sigma\tau = I\alpha$ $\omega_f - \omega_i = \alpha t$

$I_1 = MR^2$ $I_2 = \frac{1}{2}MR^2$



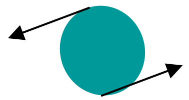
A) Solid cylinder
B) Thin cylinder shell
C) More information is needed
D) Same time required for solid and thin cylinders

Strategy to use the Newton's 2nd Law

- Draw or sketch system. Adopt coordinates, indicate rotation axes, list the known and unknown quantities, …
- Draw free body diagrams of key parts. Show forces at their points of application. find torques about a (common) axis

- May need to apply Second Law twice to each part
  - Translation: $\mathbf{F}_{net} = \sum \vec{\mathbf{F}}_i = m\vec{a}$
  - Rotation: $\vec{\tau}_{net} = \sum \vec{\tau}_i = I\vec{\alpha}$

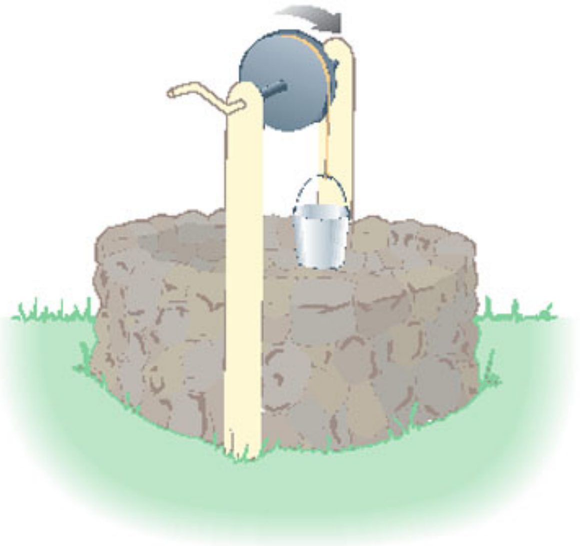
Note: can have F_{net} .eq. 0 but τ_{net} .ne. 0



- Make sure there are enough (N) equations; there may be constraint equations (extra conditions connecting unknowns)
- Simplify and solve the set of (simultaneous) equations.
- Find unknown quantities and check answers

The Falling Object

- A solid, frictionless cylindrical reel of mass $M = 2.5$ kg and radius $R = 0.2$ m is used to draw water from a well. A bucket of mass $m = 1.2$ kg is attached to a cord that is wrapped around the cylinder.
- (a) Find the tension T in the cord and acceleration a of the bucket.
- (b) If the bucket starts from rest at the top of the well and falls for 3.0 s before hitting the water, how far does it fall ?

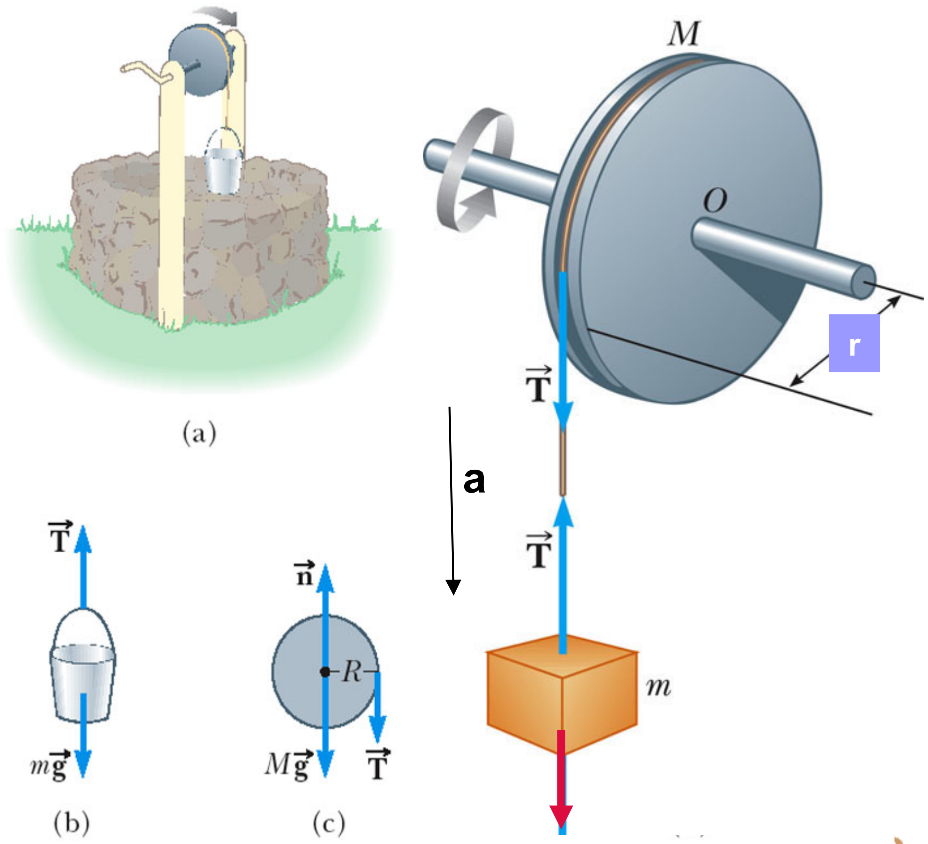


Newton 2nd Law for Rotation

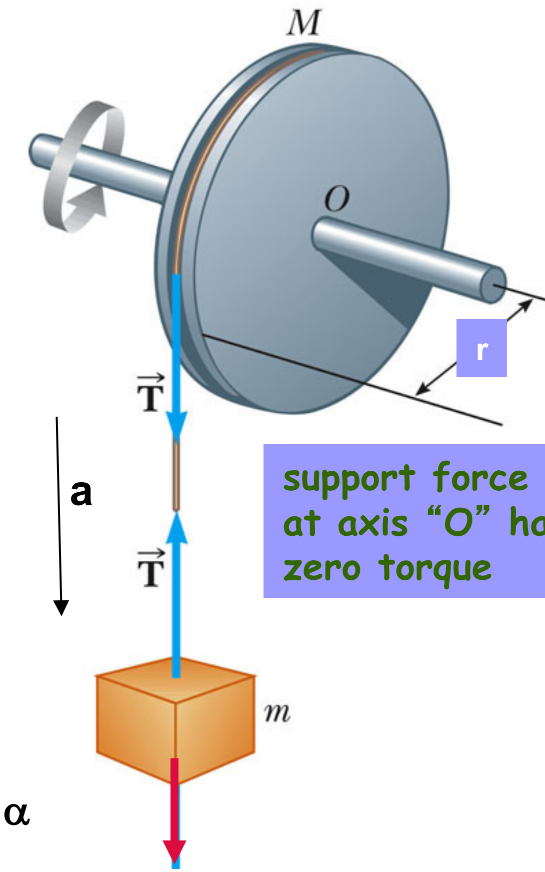
- Draw free body diagrams of each object
- Only the cylinder is rotating, so apply

  $\Sigma \tau = I \alpha$
- The bucket is falling, but not rotating, so apply

  $\Sigma F = m a$
- Remember that $a = \alpha r$ and solve the resulting equations

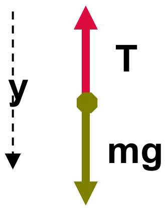


- Cord wrapped around disk, hanging weight
- Cord does not slip or stretch → constraint
- Disk's rotational inertia slows accelerations
- Let m = 1.2 kg, M = 2.5 kg, r =0.2 m



support force at axis "O" has zero torque

For mass m:

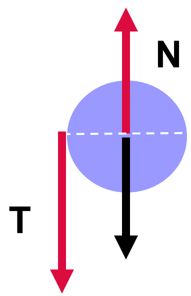


$$\sum F_y = ma = mg - T$$

$$\boxed{T = m(g-a)}$$

Unknowns: T, a

FBD for disk, with axis at "o":



$$\sum \tau_0 = +Tr = I\alpha \qquad I = \frac{1}{2}Mr^2$$

$$\boxed{\alpha = \frac{Tr}{I} = \frac{m(g-a)r}{\frac{1}{2}Mr^2}}$$

Unknowns: a, α

So far: 2 Equations, 3 unknowns →Need a constraint:

$$\boxed{a = +\alpha r}$$

from "no slipping" assumption

Substitute and solve:

$$\alpha = \frac{2mgr}{Mr^2} - \frac{2m\alpha r^2}{Mr^2} \qquad \alpha\left(1 + 2\frac{m}{M}\right) = \frac{2mg}{Mr}$$

$$\boxed{\alpha = \frac{mg}{r(m + M/2)} \quad (= 24\ \text{rad/s}^2)}$$

- Cord wrapped around disk, hanging weight
- Cord does not slip or stretch → constraint
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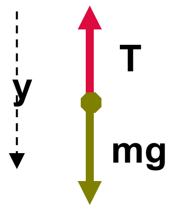
- Let m = 1.2 kg, M = 2.5 kg, r =0.2 m

For mass m:



$$\sum F_y = ma = mg - T$$

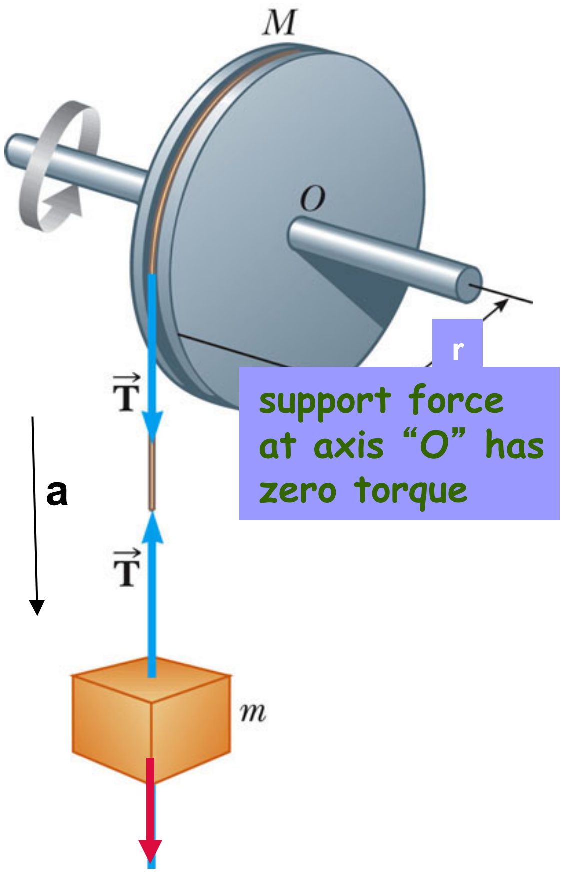
$$T = m(g-a)$$

$$\alpha = \frac{mg}{r(m+M/2)} \quad (= 24\ \text{rad/s}^2)$$

$$a = \frac{mg}{(m+M/2)} \quad (= 4.8\ \text{m/s}^2)$$

$$T = m(g-a) = 1.2(9.8-4.8) = 6\text{N}$$

$$x_f - x_f = v_i t + \frac{1}{2}at^2 = 0 + \frac{1}{2}\times 4.8 \times 3^2 = 21.6\text{m}$$



Momentum of Rotation

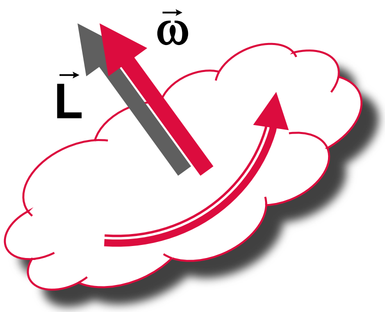
Angular Momentum

- Same basic techniques that were used in linear motion can be applied to rotational motion.
  - *F* becomes τ
  - *m* becomes I
  - *a* becomes α
  - *v* becomes ω
  - *x* becomes θ
- Linear momentum defined as $p = mv$
- What if mass of center of object is not moving, but it is rotating?
- Angular momentum $L = I\omega$

Angular Momentum of a Rigid Body

- Angular momentum of a rotating rigid object

$$\vec{L} = I\vec{\omega}$$



  - L has the same direction as ω
  - L is positive when object rotates in CCW
  - L is negative when object rotates in CW
- Angular momentum SI unit: kg m^2/s

- Calculate L of a 10 kg disc when ω = 320 rad/s, R = 9 cm = 0.09 m
- $L = I\omega$ and $I = MR^2/2$ for disc
- $L = 1/2MR^2\omega = \frac{1}{2}(10)(0.09)^2(320) = 12.96$ kg m^2/s

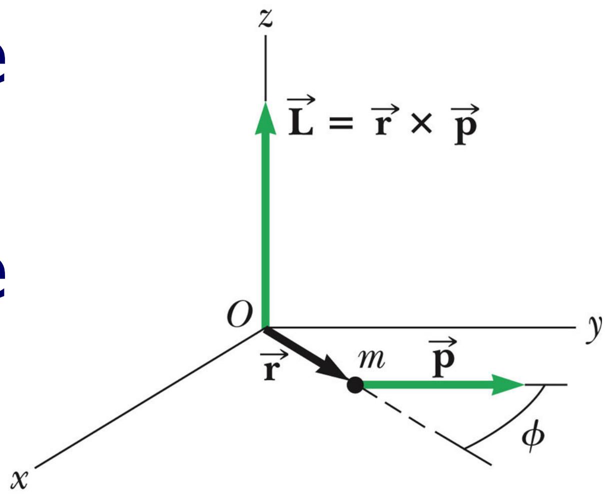
Angular Momentum of a particle

- Angular momentum of a particle

$$L = I\omega = mr^2\omega = mv_\perp r = mvr\sin\phi = rp\sin\phi$$

- Angular momentum of a particle

$$\vec{L} = \vec{r}\times\vec{p} = m(\vec{r}\times\vec{v})$$



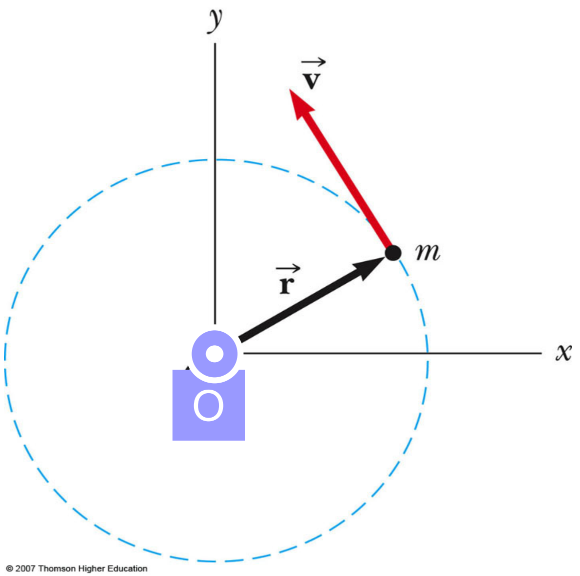
  - r is the particle's instantaneous position vector
  - p is its instantaneous linear momentum
  - Only tangential momentum component contribute
  - r and p tail to tail form a plane, L is perpendicular to this plane

Angular Momentum of a Particle in Uniform Circular Motion

Example: A particle moves in the xy plane in a circular path of radius r. Find the magnitude and direction of its angular momentum relative to an axis through O when its velocity is v.

- The angular momentum vector points out of the diagram
- The magnitude is

  $L = rp \sin\theta = mvr \sin(90^o) = mvr$
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path



Angular momentum III

- Angular momentum of a system of particles

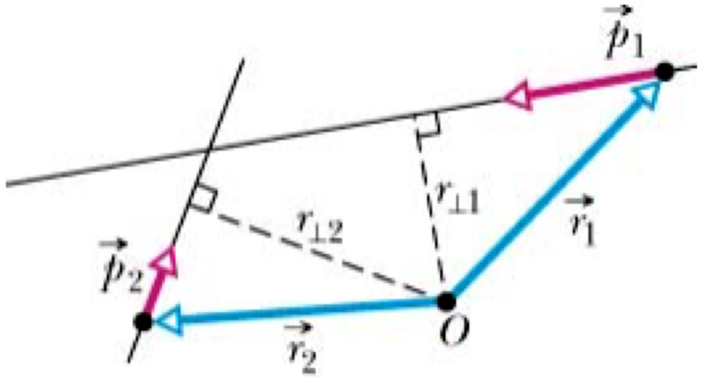
$$\vec{L}_{net} = \vec{L}_1 + \vec{L}_2 + \ldots + \vec{L}_n = \sum_{\text{all } i} \vec{L}_i = \sum_{\text{all } i} \vec{r}_i \times \vec{p}_i$$

  - angular momenta add as vectors
  - be careful of sign of each angular momentum

for this case:

$$\vec{L}_{net} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$|\vec{L}_{net}| = +r_{\perp 1} p_1 - r_{\perp 2} p_2$$



Calculating angular momentum for particles

Two objects are moving as shown in the figure. What is their total angular momentum about point O?

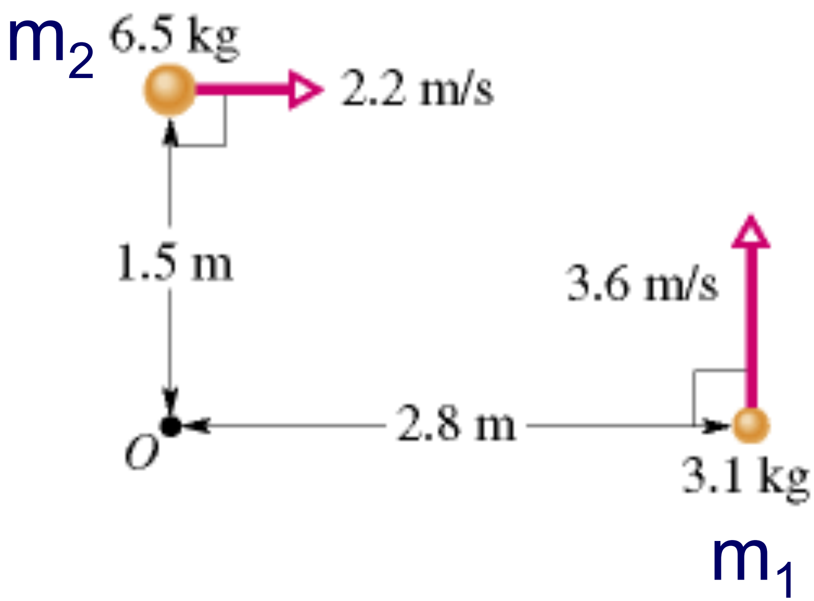
$$\vec{L}_{net} = \vec{L}_1 + \vec{L}_2 = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$$

$$L_{net} = r_1 m v_1 \sin\theta_1 - r_2 m v_2 \sin\theta_2$$

$$= r_1 m v_1 - r_2 m v_2$$

$$= 2.8 \times 3.1 \times 3.6 - 1.5 \times 6.5 \times 2.2$$

$$= 31.25 - 21.45 = 9.8\, kgm^2 / s$$



Linear Momentum and Force

- Linear motion: apply force to a mass
- The force causes the linear momentum to change
- The net force acting on a body is the time rate of change of its linear momentum

$$\vec{F}_{net} = \Sigma\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = m\vec{v}$$

$$\vec{J} = \vec{F}_{net}\Delta t = \Delta\vec{p}$$

Angular Momentum and Torque

- Net torque acting on an object is equal to the time rate of change of the object's angular momentum

$$\sum \tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = I(\frac{\omega - \omega_0}{\Delta t}) = \frac{I\omega - I\omega_0}{\Delta t}$$

- Using the definition of angular momemtum

$$\sum \tau = \frac{\text{change in angular momentum}}{\text{time interval}} = \frac{\Delta L}{\Delta t}$$

Angular Momentum and Torque

- Rotational motion: apply torque to a rigid body
- The torque causes the angular momentum to change
- The net torque acting on a body is the time rate of change of its angular momentum

$$\vec{F}_{net} = \Sigma\vec{F} = \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \vec{\tau}_{net} = \Sigma\vec{\tau} = \frac{d\vec{L}}{dt}$$

- $\Sigma\vec{\tau}$ and $\vec{L}$ to be measured about the same origin
- The origin should not be accelerating, should be an inertial frame

Isolated System

- Isolated system: net external torque acting on a system is ZERO
  - Scenario #1: no external forces
  - Scenario #2: net external force acting on a system is ZERO

$$\sum \vec{\tau}_{ext} = \frac{d\vec{L}_{tot}}{dt} = 0$$

$$\vec{L}_{tot} = \text{constant} \qquad \text{or} \qquad \vec{L}_i = \vec{L}_f$$

Conservation of Angular Momentum

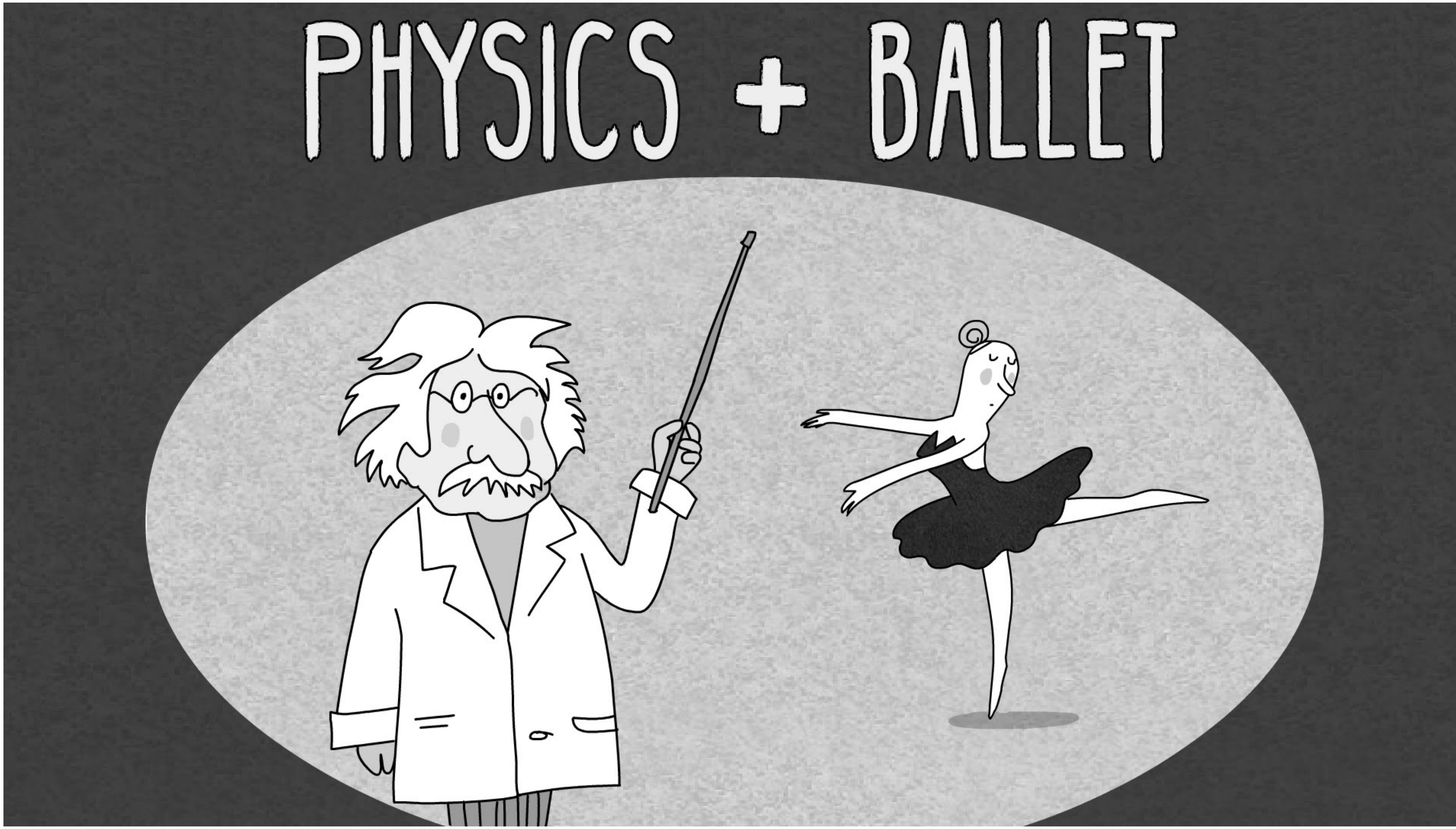
$$\vec{L}_{tot} = \text{constant} \qquad \text{or} \qquad \vec{L}_i = \vec{L}_f$$

- where *i* denotes initial state, *f* is final state
- L is conserved separately for x, y, z direction
- For an isolated system consisting of particles,

$$\vec{L}_{tot} = \sum \vec{L}_n = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \cdots = \text{constant}$$

- For an isolated system is deformable

$$I_i \omega_i = I_f \omega_f = \text{constant}$$



- One of the classic scenes in Swan Lake
- Physics explained

Isolated System

$$\vec{\tau}_{net} = 0 \quad about\ z\text{-}axis \quad \Rightarrow \quad \vec{L} = \text{constant}$$

$$\vec{L} = \sum_{initial} I_i \omega_i = \sum_{final} I_f \omega_f$$

Moment of inertia changes

How fast does the ballerina spin?

The ballerina is initially rotating with angular speed 1.2 radian/s with her arms/legs out-stretched. The moment of inertia is 6.0 kg m^2. Now she pull in her arms and legs and the moment of inertial reduces to 2.0 kg m^2.

(a) what is the resulting angular speed of the ballerina?

(b) what is the ratio of the new kinetic energy to the original kinetic energy?

L is constant… while moment of inertia changes

Larger I_i = 6 kg-m^2

Smaller ω_i = 1.2 rad/s

Smaller I_f = 2 kg-m^2

Larger ω_f = ? rad/s

L is constant… while moment of inertia changes,

$$\text{Zero external torque} \Rightarrow \mathrm{L}_{\text{final}} = \mathrm{L}_{\text{initial}} = \mathrm{L}$$

$$\text{...about a fixed axis} \quad \vec{\mathrm{L}} = I_i\omega_i = I_f\omega_f$$

Solution (a):

$$\omega_f = \frac{I_i}{I_f}\omega_i = \frac{6}{2}\times 1.2 = 3.6 \text{ rad/s}$$

Solution (b):

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}I_f\omega_f^2}{\frac{1}{2}I_i\omega_i^2} = \frac{I_f}{I_i}\left(\frac{\omega_f}{\omega_f}\right)^2 = \frac{I_f}{I_i}\left(\frac{I_i}{I_f}\right)^2 = \frac{I_i}{I_f} = 3$$

KE has increased!!

SUMMARY

Translation		Rotation	
Force	$\vec{F}$	Torque	$\vec{\tau} = \vec{r} \times \vec{F}$
Linear Momentum	$\vec{p} = m\vec{v}$	Angular Momentum	$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$
Kinetic Energy	$K = \frac{1}{2}mv^2$	Kinetic Energy	$K = \frac{1}{2}I\omega^2$

Systems and Rigid Bodies

Linear Momentum	$\vec{P} = \sum \vec{p}_i = M\vec{v}_{cm}$	Angular Momentum	$\vec{L} = \sum \vec{L}_i = \sum I_i \vec{\omega}_i$
Second Law	$\vec{F}_{net} = \frac{d\vec{P}}{dt}$	Second Law	$\vec{\tau}_{net} = \frac{d\vec{L}_{sys}}{dt}$

Momentum conservation - for closed, isolated systems

$\vec{P}_{sys} = \text{constant}$ $\vec{L}_{sys} = \text{constant}$

Apply separately to x, y, z axes