Phys 321: Lecture 5

Stellar Atmospheres

Prof. Bin Chen
Tiernan Hall 101
bin.chen@njit.edu

Radiation Field Fundamentals 1

- **Specific intensity** I_λ

Energy per unit area per unit time per unit solid angle per unit wavelength

$$I_\lambda = \frac{dE}{\cos\theta dA dt d\Omega d\lambda}$$

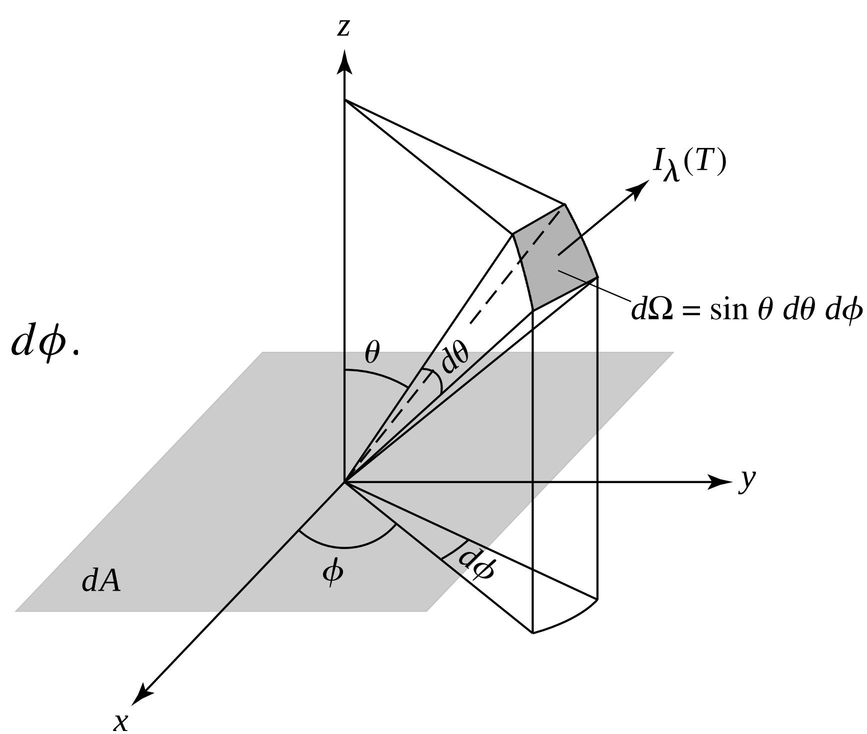
- Mean specific intensity

$$\langle I_\lambda \rangle \equiv \frac{1}{4\pi}\int I_\lambda \, d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \, \sin\theta \, d\theta \, d\phi .$$

For **isotropic** radiation field: $\langle I_\lambda \rangle = I_\lambda$

Blackbody radiation is isotropic, so

$$\langle I_\lambda \rangle = B_\lambda$$



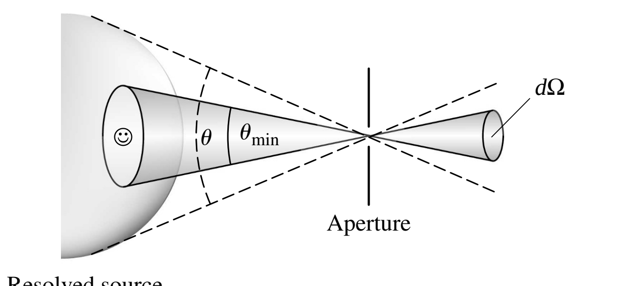
Radiation Field Fundamentals 2

- **Specific radiative flux**, or **flux density** F_λ

Energy per unit area per unit time per unit wavelength

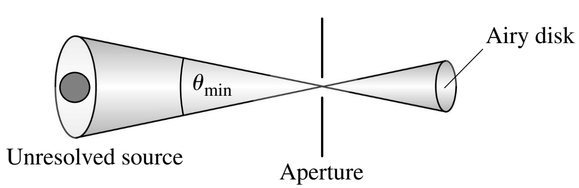
$$F_\lambda \, d\lambda = \int I_\lambda \, d\lambda \, \cos\theta \, d\Omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \, d\lambda \, \cos\theta \, \sin\theta \, d\theta \, d\phi$$

Which quantity do we measure using a telescope?



(a)

Resolved source: specific intensity



(b)

Unresolved source: radiative flux

Radiation Field Fundamentals 3

- **Specific energy density** u_λ

Energy per unit volume per unit wavelength

Energy within trap of size dL and dA:

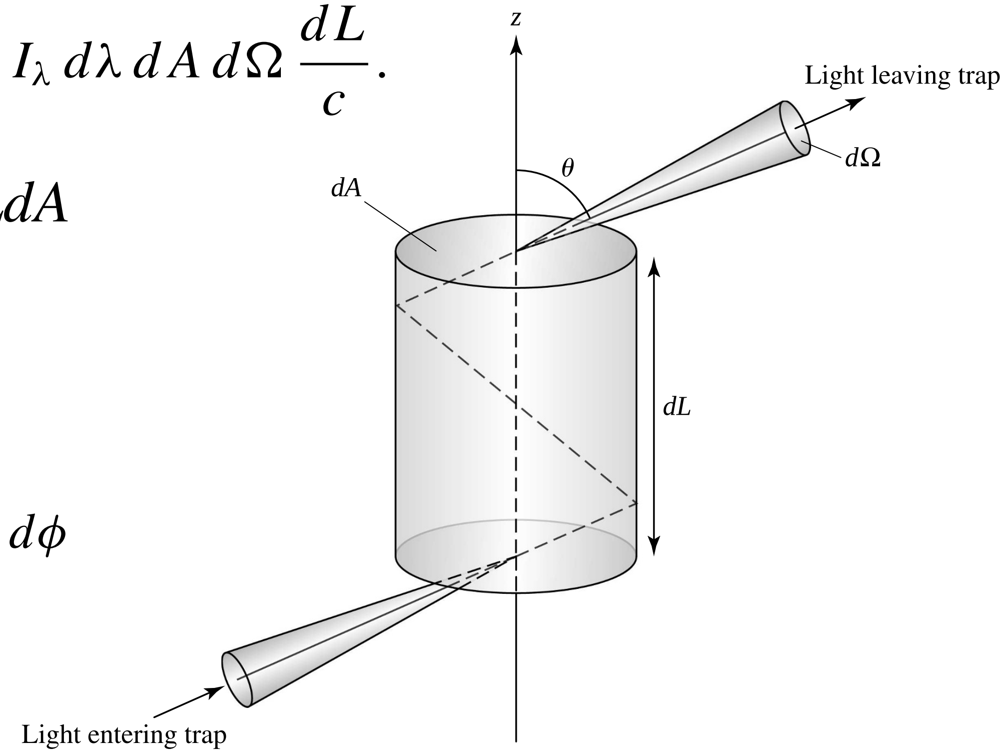
$$E_\lambda \, d\lambda = I_\lambda \, d\lambda \, dt \, dA \, \cos\theta \, d\Omega = I_\lambda \, d\lambda \, dA \, d\Omega \, \frac{dL}{c}.$$

$$u_\lambda d\lambda = E_\lambda d\lambda / dV = E_\lambda d\lambda / dLdA$$

$$u_\lambda \, d\lambda = \frac{1}{c} \int I_\lambda \, d\lambda \, d\Omega$$

$$= \frac{1}{c} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda \, d\lambda \, \sin\theta \, d\theta \, d\phi$$

$$= \frac{4\pi}{c} \langle I_\lambda \rangle \, d\lambda.$$



Radiation Field Fundamentals 3

- **Specific energy density** u_λ

For blackbody (isotropic): $$u_\lambda \, d\lambda = \frac{4\pi}{c} B_\lambda \, d\lambda = \frac{8\pi hc/\lambda^5}{e^{hc/\lambda kT} - 1} \, d\lambda$$

- Total energy density u

For blackbody (isotropic): $$u = \frac{4\pi}{c} \int_0^\infty B_\lambda(T) \, d\lambda = \frac{4\sigma T^4}{c} = aT^4.$$

where $a \equiv 4\sigma/c$ is known as the *radiation constant* and has the value

$$a = 7.565767 \times 10^{-16} \text{ J m}^{-3}\text{K}^{-4}.$$

Radiation Field Fundamentals 4

- **Radiation pressure** $P_{\mathrm{rad},\lambda}$

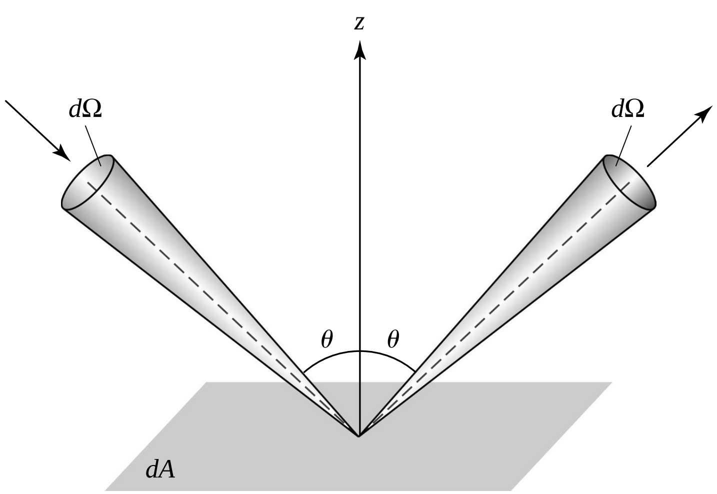
Pressure from all photons in a unit wavelength

Each photon of energy E = hν carries momentum p = E/c

Change of momentum per unit area -> force per unit area -> pressure

$$\begin{aligned} dp_\lambda\, d\lambda &= \left[(p_\lambda)_{\mathrm{final},z} - (p_\lambda)_{\mathrm{initial},z}\right] d\lambda \\ &= \left[\frac{E_\lambda\ \cos\theta}{c} - \left(-\frac{E_\lambda\ \cos\theta}{c}\right)\right] d\lambda \\ &= \frac{2\,E_\lambda\ \cos\theta}{c}\, d\lambda \\ &= \frac{2}{c}\, I_\lambda\, d\lambda\, dt\, dA\ \cos^2\theta\, d\Omega, \end{aligned}$$

For photons bouncing off a surface dA with incident angle θ



Radiation Field Fundamentals 4

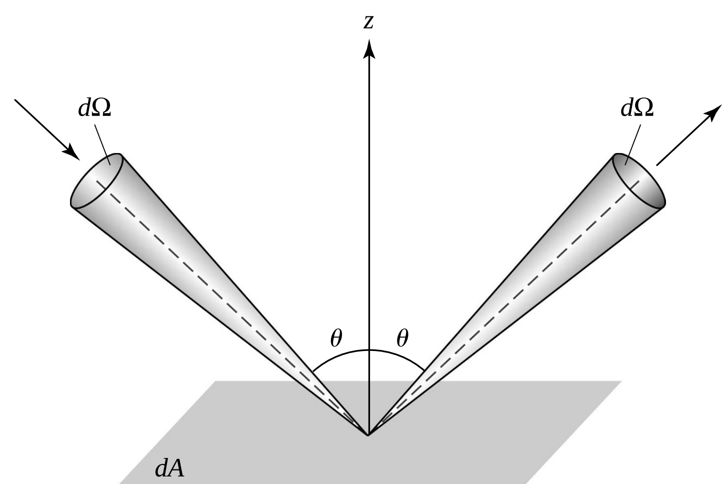
- **Radiation pressure** $P_{\mathrm{rad},\lambda}$

Pressure from all photons in a unit wavelength

Integrating over the hemisphere:

$$P_{\mathrm{rad},\lambda}\, d\lambda = \frac{2}{c}\int_{\mathrm{hemisphere}} I_\lambda\, d\lambda\, \cos^2\theta\, d\Omega \quad \text{(reflection)}$$

$$= \frac{2}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\lambda\, d\lambda\, \cos^2\theta\, \sin\theta\, d\theta\, d\phi.$$



For an isotropic "photon gas" without an actual "wall", factor 2 is removed, and we have to integrate over all solid angle

$$P_{\mathrm{rad},\lambda}\, d\lambda = \frac{1}{c}\int_{\mathrm{sphere}} I_\lambda\, d\lambda\, \cos^2\theta\, d\Omega \quad \text{(transmission)}$$

$$= \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda\, d\lambda\, \cos^2\theta\, \sin\theta\, d\theta\, d\phi$$

$$= \frac{4\pi}{3c} I_\lambda\, d\lambda \quad \text{(isotropic radiation field).}$$

Radiation Field Fundamentals 4

- **Radiation pressure** $P_{\text{rad},\lambda}$

Pressure from all photons in a unit wavelength

To obtain the **total radiation pressure,** we integrate over all wavelengths

$$P_{\text{rad}} = \int_0^\infty P_{\text{rad},\lambda}\, d\lambda.$$

For blackbody radiation

$$P_{\text{rad}} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\, d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4 = \frac{1}{3}u$$

Blackbody radiation pressure is **one-third** of the photon energy density

Compare to

Pressure of an ideal, monatomic gas is **two-thirds** of its energy density

Mean free path

In a time t, a hydrogen atom moves through a volume

$$V = \pi (2a_0)^2 vt = \sigma vt,$$

$$\sigma \equiv \pi (2a_0)^2$$

is the **collisional cross section**

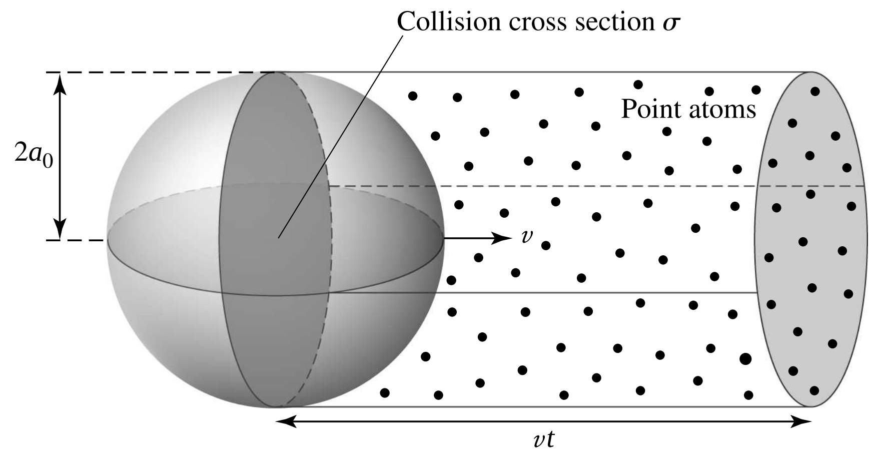


FIGURE 6 Mean free path, ℓ, of a hydrogen atom.

The averaged distance travelled between each collision

$$\ell = \frac{vt}{n\sigma vt} = \frac{1}{n\sigma}$$

This is called the **mean free path** between collisions

For hydrogen atoms in the solar photosphere

$$n = \frac{\rho}{m_H} = 1.25 \times 10^{23} \text{ m}^{-3}$$

$$\sigma = \pi (2a_0)^2 = 3.52 \times 10^{-20} \text{ m}^2.$$

$$\ell = \frac{1}{n\sigma} = 2.27 \times 10^{-4} \text{ m}$$ Very small!

Thermaldynamic Equilibrium

- A steady-state condition with no net flow of energy within the volume or between the matter (particles) and the radiation field.
  - Can be described by a single temperature
  - NOT the case for stars!
- However, if the distance over which the temperature changes is significantly larger than the mean free path of particles and photons, within a limited volume, it achieves the **local thermodynamic equilibrium (LTE)** – can be described by a single temperature
- E.g., the temperature scale height of solar photosphere

$$H_T \equiv \frac{T}{|dT/dr|} = \frac{5685\ \mathrm{K}}{(5790\ \mathrm{K} - 5580\ \mathrm{K})/(25.0\ \mathrm{km})} = 677\ \mathrm{km}$$

Opacity

A measure of the absorption of light

Distance traveled **from** the light source

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda \, ds.$$

Opacity or absorption coefficient. Unit: $m^2\ kg^{-1}$

Light traveling through an absorbing gas

$$\int_{I_{\lambda,0}}^{I_{\lambda,f}} \frac{dI_\lambda}{I_\lambda} = -\int_0^s \kappa_\lambda \rho \, ds$$

Which leads to

$$I_\lambda = I_{\lambda,0} e^{-\int_0^s \kappa_\lambda \rho \, ds}$$

For constant opacity and density

$$I_\lambda = I_{\lambda,0} e^{-\kappa_\lambda \rho s}.$$

$I_\lambda = I_{\lambda,0} e^{-s/l}$ Where $l = 1/k_\lambda \rho = 1/n\sigma_\lambda$ is the **mean free path of the photons**

Opacity and mean free path of photons

In solar photosphere

$$\rho = 2.1 \times 10^{-4}\ \mathrm{kg\ m^{-3}},$$

Opacity for 500 nm photons:

$$\kappa_{500} = 0.03\ \mathrm{m^2\ kg^{-1}}$$

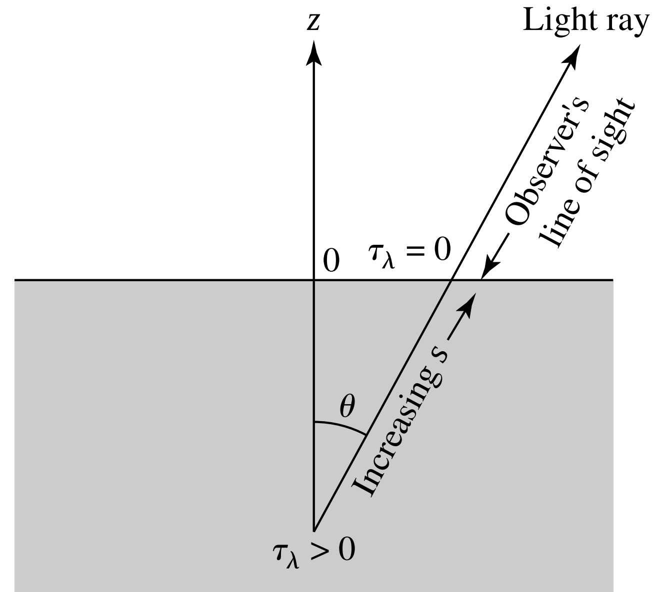
Mean free path for 500 nm photons:

$$\ell = \frac{1}{\kappa_{500}\rho} = 160\ \mathrm{km}.$$

Comparable to temperature scale height (677 km)! LTE?

Optical Depth

$$d\tau_\lambda = -\kappa_\lambda \rho \, ds,$$



Defined **back along a light ray**

Difference in optical depth from initial position (s = 0) and final position (s = s_f):

$$\Delta\tau_\lambda = \tau_{\lambda,f} - \tau_{\lambda,0} = -\int_0^s \kappa_\lambda \rho \, ds$$

=0

At position s: $\tau_\lambda = \int_0^s \kappa_\lambda \rho \, ds$

Optical depth

Intensity of a ray traveling through a gas from an optical depth τ_λ

$$I_\lambda = I_{\lambda,0} e^{-\int_0^s \kappa_\lambda \rho \, ds}$$

$$I_\lambda = I_{\lambda,0} e^{-\tau_\lambda}$$

Comparing to $I_\lambda = I_{\lambda,0} e^{-s/l}$

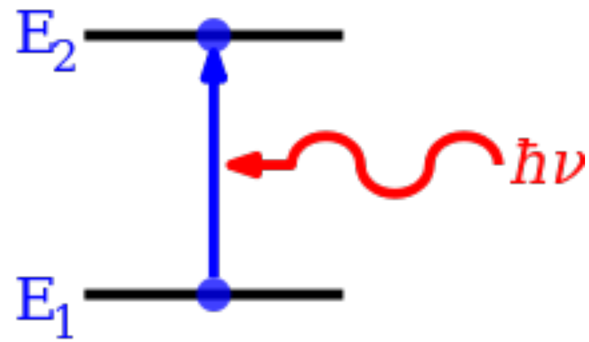
Optical depth can be thought of as the **number of mean free paths** from the original position to the surface, as measured along the ray's path

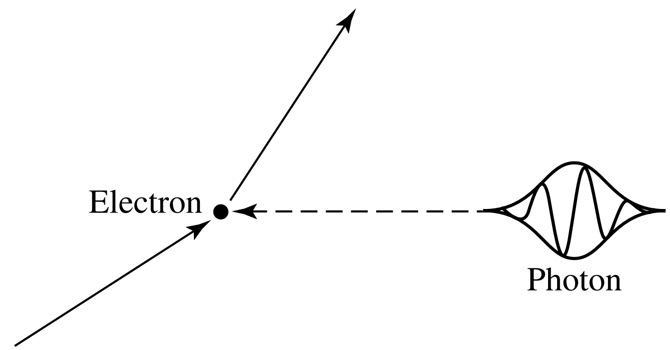
$\tau_\lambda \gg 1$: Optically thick

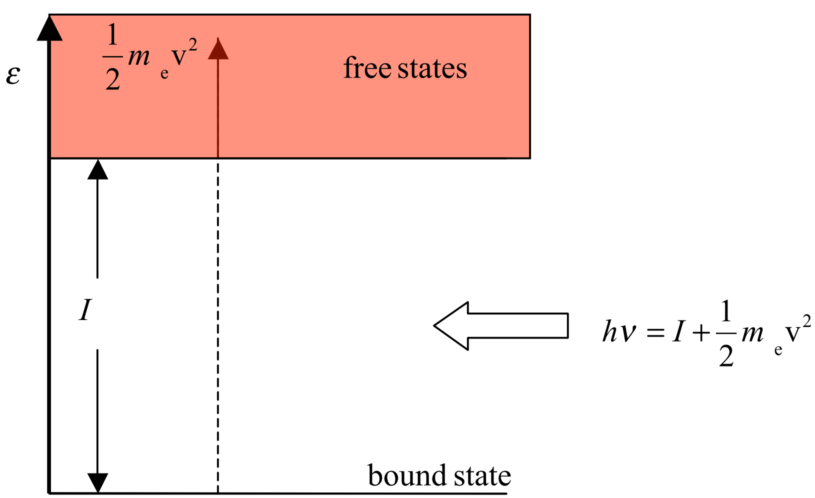
$\tau_\lambda \ll 1$: Optically thin

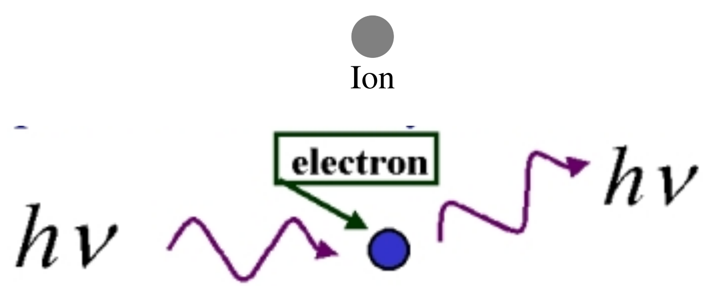
General sources of opacity

- Bound-bound transitions
- Bound-free absorption
- Free-free absorption
- Electron scattering









Rosseland mean opacity

- Mean opacity averaged over all wavelengths

$$\frac{1}{\overline{\kappa}} \equiv \frac{\displaystyle\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T}\, d\nu}{\displaystyle\int_0^\infty \frac{\partial B_\nu(T)}{\partial T}\, d\nu}.$$

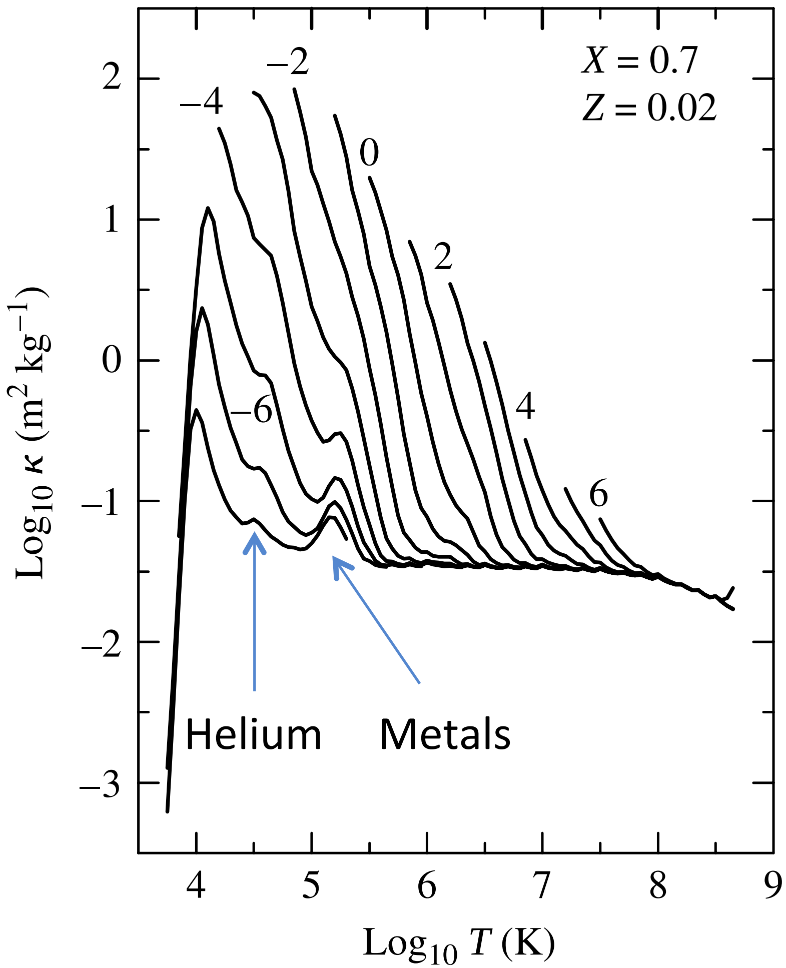
Bound-free $\qquad \overline{\kappa}_{\rm bf} = 4.34 \times 10^{21} \frac{g_{\rm bf}}{t} Z(1+X) \frac{\rho}{T^{3.5}} \text{ m}^2 \text{ kg}^{-1}$

Free-free $\qquad \overline{\kappa}_{\rm ff} = 3.68 \times 10^{18}\, g_{\rm ff}\, (1-Z)(1+X) \frac{\rho}{T^{3.5}} \text{ m}^2 \text{ kg}^{-1}$

Electron-scattering $\qquad \overline{\kappa}_{\rm es} = 0.02(1+X) \text{ m}^2 \text{ kg}^{-1}$

Total Rosseland mean opacity $\qquad \overline{\kappa} = \overline{\kappa_{\rm bb} + \kappa_{\rm bf} + \kappa_{\rm ff} + \kappa_{\rm es} + \kappa_{\rm H^-}}.$

Rosseland mean opacity



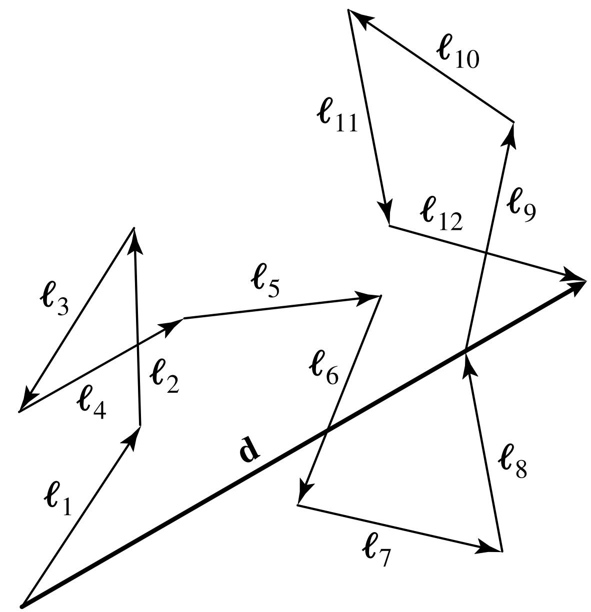
Rosseland mean opacity for a composition that is 70% hydrogen, 28% helium, and 2% metals by mass. Labeled are logarithmic value of the density

Radiative Transfer: Random walk

Displacement after N random steps: $\mathbf{d} = \boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 + \boldsymbol{\ell}_3 + \cdots + \boldsymbol{\ell}_N$

Take a vector dot of itself:

$$\begin{aligned}\mathbf{d}\cdot\mathbf{d} &= \boldsymbol{\ell}_1\cdot\boldsymbol{\ell}_1 + \boldsymbol{\ell}_1\cdot\boldsymbol{\ell}_2 + \cdots + \boldsymbol{\ell}_1\cdot\boldsymbol{\ell}_N \\ &\quad + \boldsymbol{\ell}_2\cdot\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2\cdot\boldsymbol{\ell}_2 + \cdots + \boldsymbol{\ell}_2\cdot\boldsymbol{\ell}_N \\ &\quad + \cdots + \boldsymbol{\ell}_N\cdot\boldsymbol{\ell}_1 + \boldsymbol{\ell}_N\cdot\boldsymbol{\ell}_2 + \cdots + \boldsymbol{\ell}_N\cdot\boldsymbol{\ell}_N \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\ell}_i\cdot\boldsymbol{\ell}_j,\end{aligned}$$



Which becomes

$$\begin{aligned}d^2 &= N\ell^2 + \ell^2[\cos\theta_{12} + \cos\theta_{13} + \cdots + \cos\theta_{1N} \\ &\quad + \cos\theta_{21} + \cos\theta_{23} + \cdots + \cos\theta_{2N} \\ &\quad + \cdots + \cos\theta_{N1} + \cos\theta_{N2} + \cdots + \cos\theta_{N(N-1)}] \\ &= N\ell^2 + \ell^2\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\cos\theta_{ij},\end{aligned}$$

$$d = \ell\sqrt{N}$$

Radiative Transfer: Random walk

Distance traveled after N steps: $d = \ell\sqrt{N}$

Distance traveled from optical depth τ_λ to surface: $d = \tau_\lambda \ell = \ell\sqrt{N}$

Number of steps to reach surface: $N = \tau_\lambda^2$

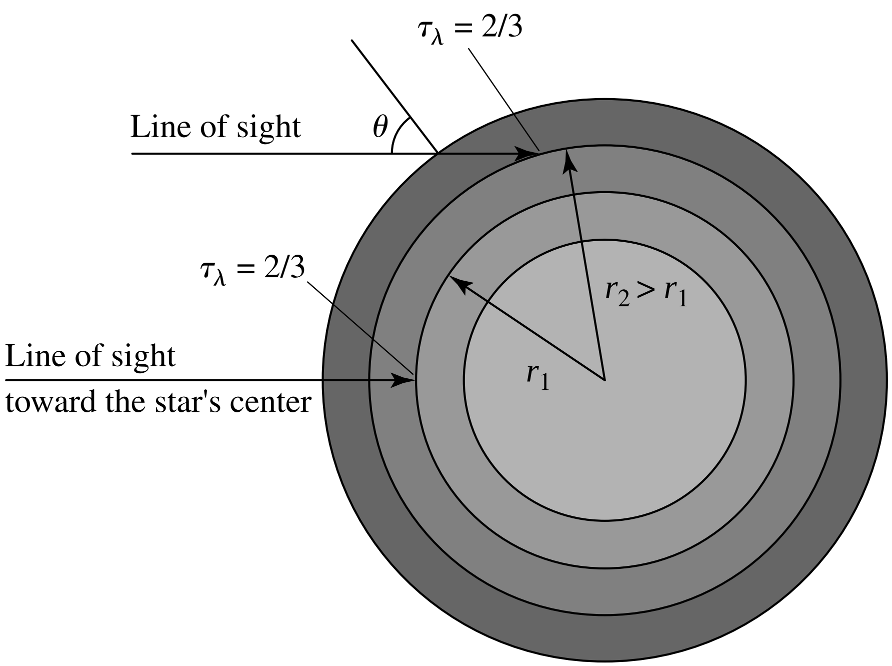
At the $\tau_\lambda \approx 1$ layer and above, photons can escape easily
-> the stellar atmosphere only becomes opaque below this level

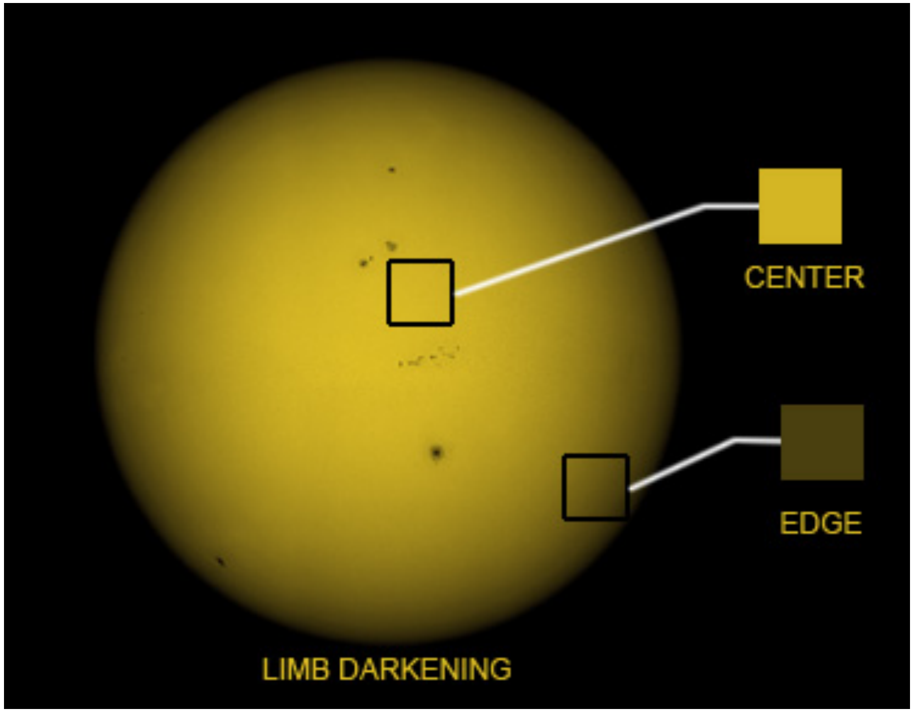
An important implication:
-> Looking into a star at any angle, we always look back to an optical depth of about $\tau_\lambda = 2/3$
(from more careful analysis), as measured straight back along the line of sight

Stellar photosphere is defined as the layer from which its visible light originates, that is

$$\tau_\lambda = 2/3$$

Radiative Transfer: Limb darkening





The visible Sun

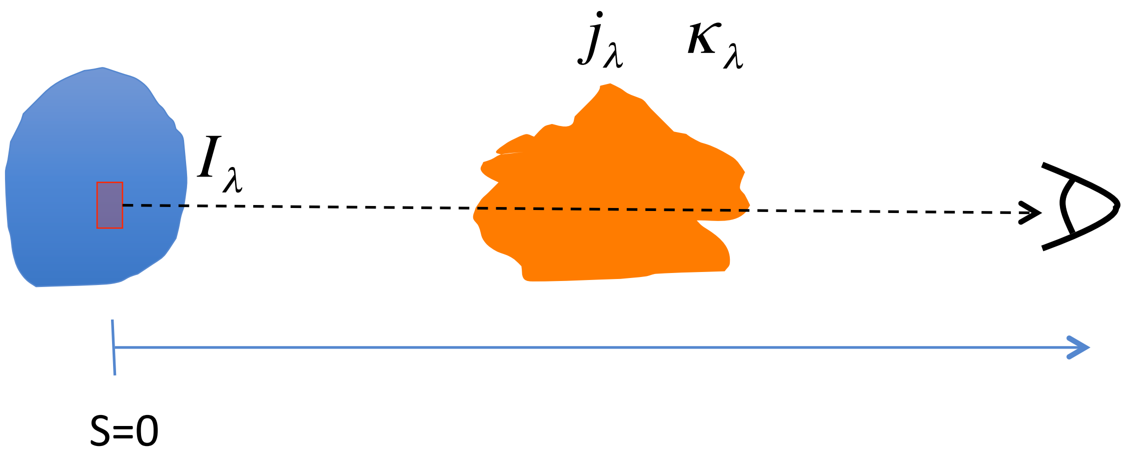
Radiative Transfer Equation: The Emission Coefficient

- Previously we have been dealing with the **absorption** of light along its path
- The material along the path can also emit photons, and adds up to its intensity
- We use the **emission coefficient** to describe this effect. For pure emission (no absorption)

$$dI_\lambda = j_\lambda \rho \, ds$$

Emission Coefficient. Unit: m s^{-3} sr^{-1}

Radiative Transfer Equation



$$dI_\lambda = -\kappa_\lambda \rho I_\lambda \, ds + j_\lambda \rho \, ds$$

Divide both sides by $-\kappa_\lambda \rho \, ds$:

$S_\lambda \equiv j_\lambda / \kappa_\lambda$ is the **source function**

$$-\frac{1}{\kappa_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - \frac{j_\lambda}{\kappa_\lambda}$$

or

$$-\frac{1}{\kappa_\lambda \rho} \frac{dI_\lambda}{ds} = I_\lambda - S_\lambda .$$

Radiative Transfer Equation

$$-\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - S_\lambda.$$

or

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

$S_\lambda \equiv j_\lambda/\kappa_\lambda$ is the **source function**

with $d\tau_\lambda = -\kappa_\lambda \rho\, ds,$

Implications:

- Left-hand side = 0: $I_\lambda = S_\lambda$ -> intensity ***equals*** to source function
- $I_\lambda > S_\lambda$: $dI_\lambda / ds < 0$ -> intensity ***decreases*** with distance
- $I_\lambda < S_\lambda$: $dI_\lambda / ds > 0$ -> intensity ***increases*** with distance

The intensity of the light ***tries to become equal*** to the local value of the source function

Radiative Transfer Equation: Blackbody radiation

$$-\frac{1}{\kappa_\lambda \rho}\frac{dI_\lambda}{ds} = I_\lambda - S_\lambda.$$

or

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

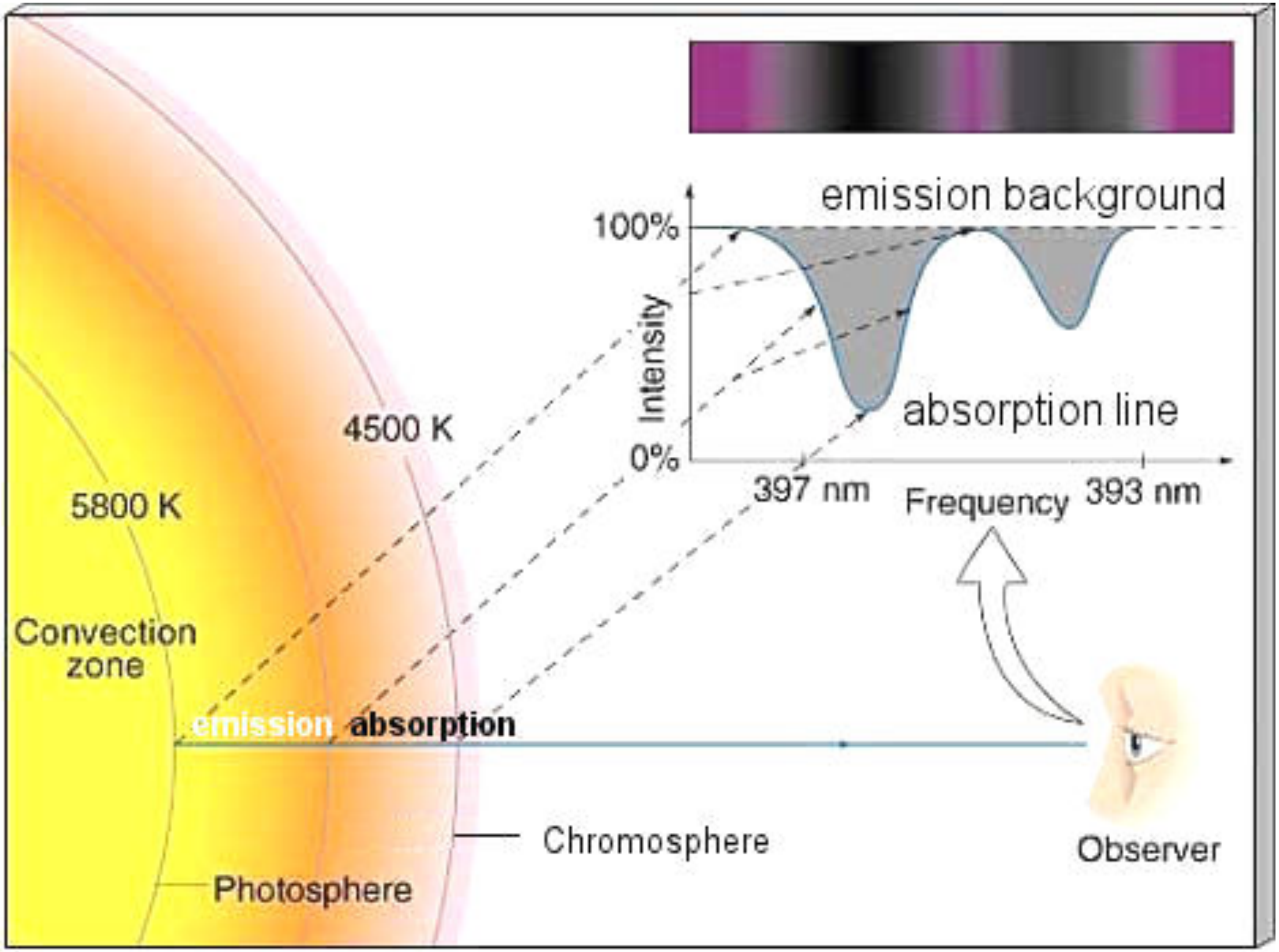
Blackbody radiation:

- emission is balanced by absorption, no net gain or loss of intensity along the light path $dI_\lambda / ds = 0$
- Intensity $I_\lambda = B_\lambda$
- So

$$S_\lambda = B_\lambda$$

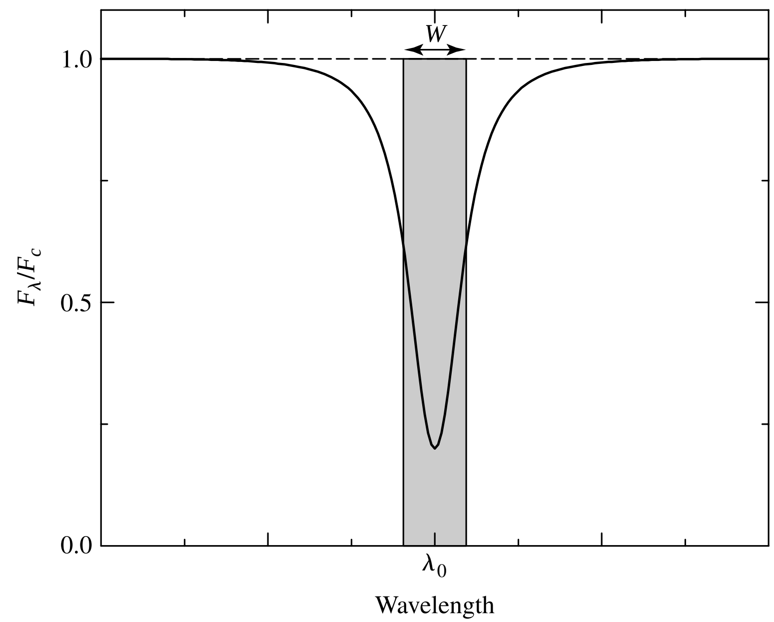
- The source function of a blackbody is equal to the Planck function
- This also applies to **local thermodynamic equilibrium (LTE)**. However, in LTE, not necessarily $I_\lambda = B_\lambda$ unless τ_λ much greater than 1.

Formation of absorption lines



The Profiles of Spectral Lines

- Absorption lines are common features in stellar spectra
  - Usually they are also not entirely "black", with a "depth" of $(F_c - F_\lambda)/F_c$
  - They are also not infinitely thin
- We use equivalent width



$$W = \int \frac{F_c - F_\lambda}{F_c}\, d\lambda$$

to quantify the line "strength"

Spectral line broadening

- **Natural broadening**
  - Lifetime of an atom at a certain orbit is not infinitely long. Heisenberg's principle says the energy can not be perfectly precise at E, but with a finite width in energy

$$\Delta E \approx \frac{\hbar}{\Delta t}.$$

Lifetime of the atom in this orbit

Since $E_{\text{photon}} = hc/\lambda = E_f - E_i$

$$\Delta\lambda \approx \frac{\lambda^2}{2\pi c}\left(\frac{1}{\Delta t_i} + \frac{1}{\Delta t_f}\right)$$

You did this in problem set #2

More involved calculation gives

$$(\Delta\lambda)_{1/2} = \frac{\lambda^2}{\pi c}\frac{1}{\Delta t_0}$$

Average waiting time for the transition to occur

Full width at half maximum of the line profile

Spectral line broadening

- **Doppler Broadening**

This is due to the Doppler Effect from the thermal and/or turbulent motion of atoms in stellar atmosphere

Most probable speed in Maxwell-Boltzmann distribution:

$$\Delta\lambda/\lambda = \pm |v_r|/c.$$

$$v_{\mathrm{mp}} = \sqrt{2kT/m}.$$

So $$\Delta\lambda \approx \frac{2\lambda}{c}\sqrt{\frac{2kT}{m}}$$

More in-depth analysis gives

$$(\Delta\lambda)_{1/2} = \frac{2\lambda}{c}\sqrt{\frac{2kT\ln 2}{m}}$$

With turbulence

$$(\Delta\lambda)_{1/2} = \frac{2\lambda}{c}\sqrt{\left(\frac{2kT}{m} + v_{\mathrm{turb}}^2\right)\ln 2}.$$

Spectral line broadening

- **Pressure (and collisional) broadening**

The orbitals of an atom can be perturbed in a collision with another atom or ion

Calculation quite complicated. Just as an estimate, replace Δt_0 with the time between collisions

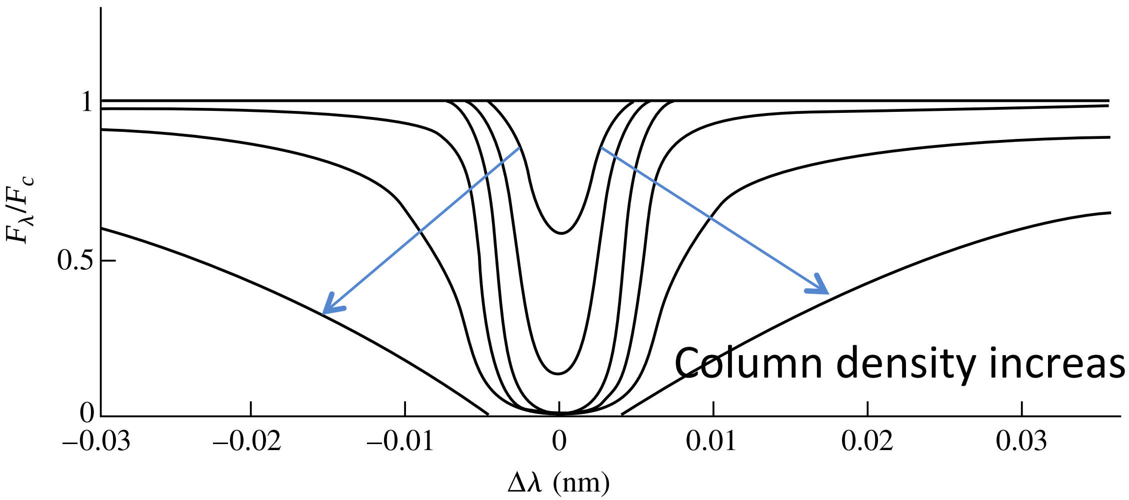
$$\Delta t_0 \approx \frac{\ell}{v} = \frac{1}{n\sigma\sqrt{2kT/m}}$$

So

$$\Delta\lambda = \frac{\lambda^2}{c}\,\frac{1}{\pi\,\Delta t_0} \approx \frac{\lambda^2}{c}\,\frac{n\sigma}{\pi}\sqrt{\frac{2kT}{m}}.$$

Width is proportional to density. Recall luminosity class of stars. Supergiant stars have very low density, while dwarf stars have greater density and appreciable broadening.

The curve of growth



Useful for estimating the number of absorbing atoms

- Initially the line grows deeper, and the equivalent width grows linearly with N
- Ultimately the core of the line saturates. At this point the wings continue to grow deeper and broader, but the core gets flatter. The equivalent width grows more slowly.
- Even denser, wings grow deeper, pressure-broadening more important, equivalent width grows more quickly again, but not as quickly as at first.