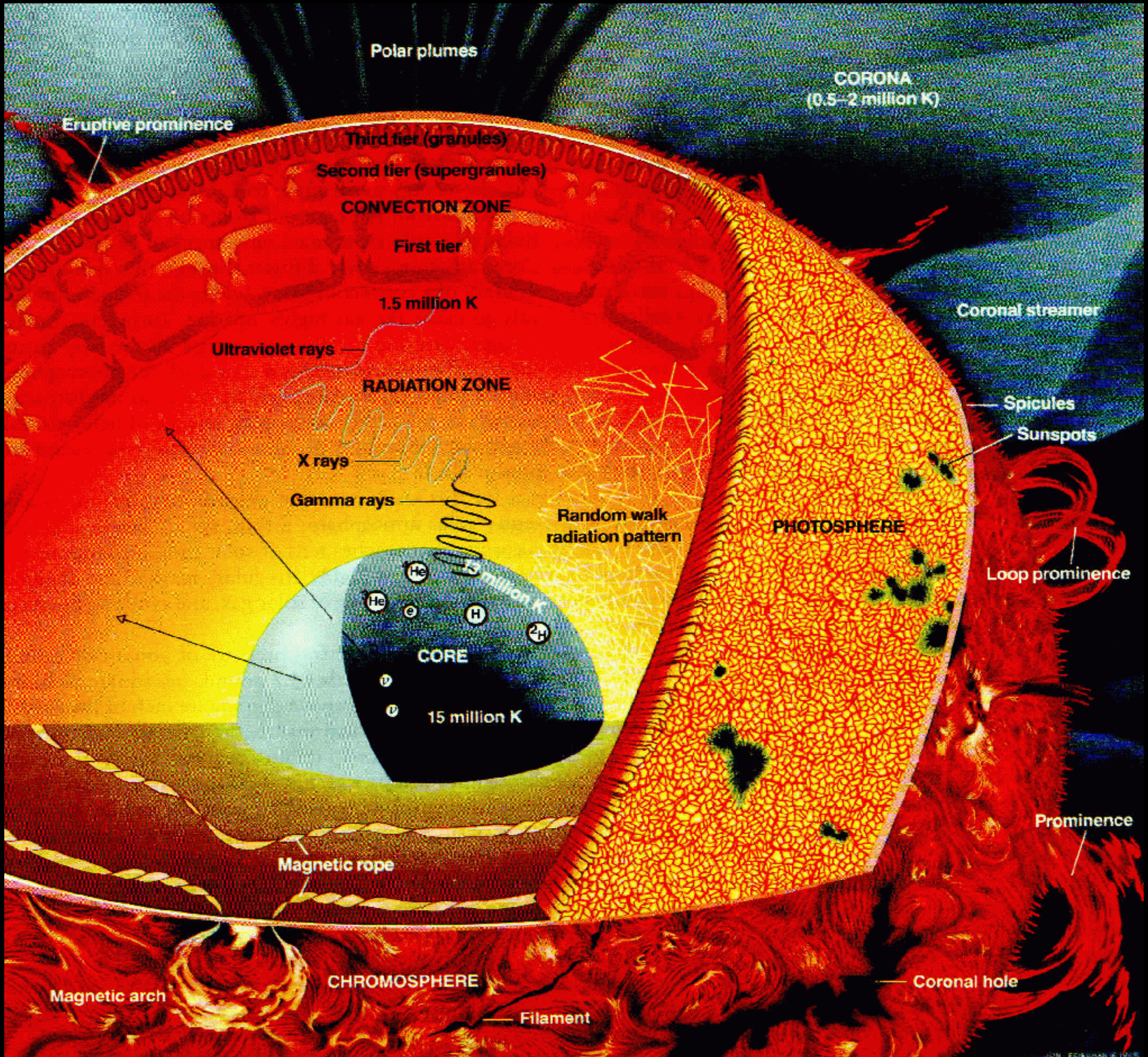
Phys 321: Lecture 5

Stellar Interiors



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Stellar Interiors

- Most of the stellar material is in the $\tau >> 1$ (or completely opaque) portion − only neutrinos can escape freely
  - Governed by basic physics
  - Difficult to observe
  - Great success in theories and computer modeling
- To understand why stars differ from one another in different parts of the H-R diagram, and subsequently their evolution, we have to learn their **internal structure**

Outstanding questions

- What supports the stars (from collapsing)?
- What powers the stars?
- What determines the internal structure of the stars?

What are stars made of?
Or “Stellar Composition”

- Stars are basically a giant sphere of hot gas made of relatively simple stuff
- Our Sun: 73% H, 26% Helium, and 1% of higher atomic number atoms (called “*metals*”)
- We write these quantities in terms of *mass fraction*

– $X = m_H n_H/\rho$ = density of hydrogen / total density

– $Y = m_{He} n_{He}/\rho$ = density of helium / total density

– $Z = m_Z n_Z/\rho$ = density of metals / total density. Also known as “*metallicity*”

Mean molecular weight

- Mean molecular weight: the *average* mass of a free particle in the gas, in units of the mass of hydrogen

$$\mu \equiv \frac{\overline{m}}{m_H} \quad \text{where} \quad \overline{m} = \frac{\rho}{n_{total}} = \frac{\text{Mass per unit volume}}{\text{Total number of particles per unit volume}}$$

$$n_{total} = n_H + n_{He} + n_Z + n_e$$

What is the mean molecular weight for fully ionized hydrogen gas?

$$n_{total} = n_H + n_e = 2n_H \qquad \rho = m_H n_H + m_e n_e \approx m_H n_H$$

$$\overline{m} = \frac{\rho}{n_{total}} \approx \frac{m_H n_H}{2n_H} = 0.5 m_H \qquad \mu = \frac{m}{m_H} \approx \frac{0.5 m_H}{m_H} = 0.5$$

What is the mean molecular weight of a fully ionized helium gas?

A. 1/2
B. 4/3
C. 1
D. 2
E. 3/4

Mean molecular weight

- Mean molecular weight of full ionized gas

$$\frac{1}{\mu} \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

- For the Sun, X = 0.73, Y = 0.26, Z = 0.01, so $1/\mu \approx 1.67$

Why do stars not collapse?

- Gravity pulls everything in. If nothing is resisting, stars collapse!
- Internal pressure!
- Most stars are in a quasi-static state: hydrostatic equilibrium

Hydrostatic Equilibrium

Newton's 2nd law:

Acceleration * dm = Net force on the cylinder

$$dm\,\frac{d^2r}{dt^2} = F_g + F_{P,t} + F_{P,b}$$

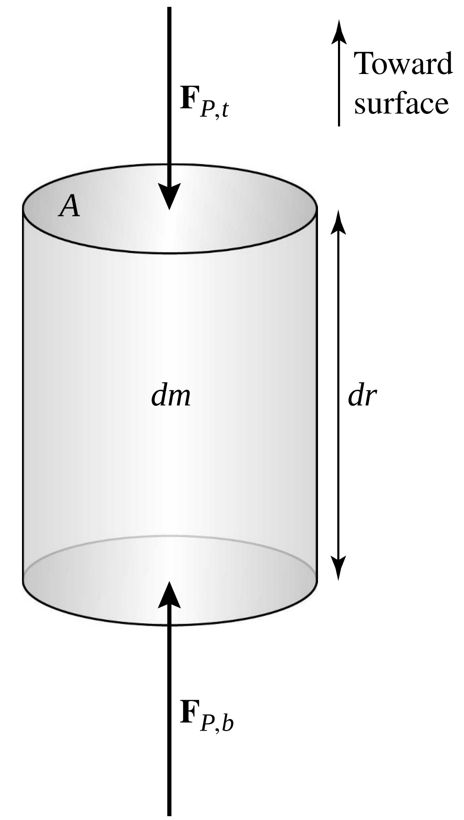
with $$F_{P,t} = -\left(F_{P,b} + dF_P\right)$$

$$dm\,\frac{d^2r}{dt^2} = F_g - dF_P$$

With gravity force $$F_g = -G\frac{M_r\,dm}{r^2}$$

And definition of pressure $$P \equiv \frac{F}{A} \qquad dF_P = A\,dP$$

We have $$dm\frac{d^2r}{dt^2} = -G\frac{M_r\,dm}{r^2} - A\,dP.$$



Hydrostatic Equilibrium

$$dm \frac{d^2 r}{dt^2} = -G \frac{M_r \, dm}{r^2} - A \, dP.$$

Now express the mass as $dm = \rho A \, dr.$

$$\rho A \, dr \frac{d^2 r}{dt^2} = -G \frac{M_r \rho A \, dr}{r^2} - A \, dP$$

$$\rho \frac{d^2 r}{dt^2} = -G \frac{M_r \rho}{r^2} - \frac{dP}{dr}$$

For static star, the left-hand side of the equation is zero. So

$$\boxed{\frac{dP}{dr} = -G \frac{M_r \rho}{r^2} = -\rho g,}$$

Where $g \equiv G M_r / r^2$ is the local acceleration of gravity at radius r

Hydrostatic Equilibrium

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2} = -\rho g,$$

- In order to have a static star, the *pressure gradient* must exist to counteract the force of gravity
- The pressure must decrease with increasing radius: larger in the interior than it is near the surface

Pressure equation of state

- What is the origin of the pressure gradient?
- We need a pressure **equation of state** of the material
- Ideal gas law: $PV = NkT$

Volume (V) # of particles (N) Temperature (T)

Or in terms of number density n: $P_g = nkT$

Using mean molecular weight

$$P_g = \frac{\rho k T}{\mu m_H}.$$

Special case: Isothermal gas

Hydrostatic equilibrium:

$$\frac{dP}{dr} = -G\frac{M_r\rho}{r^2} = -\rho g,$$

Ideal gas equation of state:

$$P_g = \frac{\rho kT}{\mu m_H}.$$

If T independent of r (isothermal):

$$\frac{dP}{dr} = \frac{kT}{\mu m_H}\frac{d\rho}{dr} = -\rho g$$

Which is a first-order differential equation with a simple solution:

$$\rho(r) = \rho_0 \exp(-r/H)$$

Where $H = kT/\mu m_H g$ is the ***density scale height***

Base density at r = 0

Similar relation holds for pressure as well

Density scale height

$$\rho(r) = \rho_0 \exp(-r / H) \qquad H = kT / \mu m_H g$$

- Esp. useful in describing the stellar atmosphere (not the interior, unfortunately)
- Pressure and density both drop by 1/e=0.368 for each increase in height of a distance H
- Scale height determines the “thickness” of the atmosphere
- Hotter atmosphere has a larger scale height
- Larger gravity -> smaller scale height

Example: Earth's atmosphere

$$H = kT / \mu m_H g$$

At the Earth's surface, T ~ 300 K, g = 9.8 m/s2, but what is μ?

Earth is 80% molecular nitrogen, N_2

μ ≈ 28

$$H = kT / \mu m_H g = (1.38 \times 10^{-23} J / K)(300K) / [28(1.67 \times 10^{-27} kg)(9.8m / s^2)]$$

≈ 9.0 km

See homework assignment for scale height in the Sun's photosphere

Other equations of state

- Ideal gas law is pretty good for stellar atmosphere and outer layers of most stars (well, "normal" stars)
- In the stellar interiors, new effects can come into play
- From lecture 5, photons have momentum $p_\gamma = h\nu/c$, and light itself can exert pressure, called *radiation pressure*

$$P_{\text{rad}} = \frac{1}{3}aT^4,$$

- Combining both ideal gas and radiation pressure terms

$$P_t = \frac{\rho k T}{\mu m_H} + \frac{1}{3}aT^4.$$

Other equations of state

- Ideal gas law is based on the classical Maxwell-Boltzmann statistics
- The equation of state will be very different when relativistic and quantum effects kick in
- Important in extremely dense stars like white dwarfs and neutrons stars
- **Fermi-Dirac statistics** for **fermions**
- **Bose-Einstein statistics** for **bosons**

Outstanding questions

- What supports the stars (from collapsing)?
- What powers the stars?
- What determines the internal structure of the stars?

Stellar Energy Sources

- Chemical energy
- Gravitational energy
- Nuclear energy
- A key to the evaluation comes from two observables
  - Solar luminosity: $L = 3.8 \times 10^{26}$ J/s
  - Age of the Sun/solar system:
    ~4.6 billion years, or $\sim 1.45 \times 10^{17}$ s
  - Total energy available should be at least:
    $E > L * t \sim 5.5 \times 10^{43}$ J

Chemical Energy

- A misconception: the Sun is a “burning fireball”
- Burning is a chemical process
  - Releases energy by rearrangement of chemical bonds, which involves bound electrons
- If every atom in the Sun were available to release, say, 10 eV of energy, how much total energy would that be?
  - Total number of hydrogen atoms: $N = M_{sun}/M_H = 1.989 \times 10^{30}$ kg / 1.67×10^{-27} kg ~ 10^{57}
  - Each atom can release 10 eV = 1.6×10^{-18} J
  - Total energy available is $10^{57} \times 1.6 \times 10^{-18} \sim 1.6 \times 10^{39}$ J
  - How long does it last? $t = E/L = 1.6 \times 10^{39}$ J / 3.8×10^{26} J/s ~ 5×10^{12} s ~ 170,000 years

Gravitational Energy

- Star can contracts over time -> reduce its gravitational potential -> releases gravitational potential energy
- Is the gravitational potential energy enough to power the stars?

Gravitation Potential Energy

Gravitational potential energy of a system of two particles

$$U = -G\frac{Mm}{r}$$

Gravitational force on a point mass dm_i located outside of a spherically symmetric mass M_r

$$dF_{g,i} = G\frac{M_r\,dm_i}{r^2}$$

The corresponding gravitational potential energy of the point mass is

$$dU_{g,i} = -G\frac{M_r\,dm_i}{r}$$

Gravitational Potential Energy

Now consider point masses are distributed uniformly within a shell of thickness dr and mass dm outside a spherically symmetric mass M_r

$$dm = 4\pi r^2 \rho \, dr,$$

The corresponding gravitational potential energy of this shell is

$$dU_g = -G \frac{M_r 4\pi r^2 \rho}{r} \, dr$$

Integrating over all mass shells from the center of the star to the surface, the total gravitational potential energy is

$$U_g = -4\pi G \int_0^R M_r \rho r \, dr.$$

Gravitational Potential Energy

An exact calculation requires knowledge of how ρ and subsequently M_r depend on r. For simplicity, let's assume ρ is constant and equal to its average value:

$$\rho \sim \overline{\rho} = \frac{M}{\frac{4}{3}\pi R^3}$$

M is the total mass of the star.

Now we may approximate M_r as

$$M_r \sim \frac{4}{3}\pi r^3 \overline{\rho}.$$

Plug in the integral

$$U_g = -4\pi G \int_0^R M_r \rho r \, dr.$$

We have

$$U_g \sim -\frac{16\pi^2}{15} G \overline{\rho}^2 R^5 \sim -\frac{3}{5}\frac{GM^2}{R}.$$

Gravitational Potential Energy

- Applying the virial theorem: When a system is in one equilibrium state and changes to another equilibrium state, the difference in potential energy goes equally into 1) the mechanical energy of the system and 2) energy loss via radiation or other loss mechanisms.

Total potential energy: $$U_g \sim -\frac{16\pi^2}{15} G\overline{\rho}^2 R^5 \sim -\frac{3}{5}\frac{GM^2}{R}$$

Total mechanical energy of the star: $$E \sim -\frac{3}{10}\frac{GM^2}{R}$$

Kevin-Helmholtz Timescale

- If the Sun were originally much larger than it is today, how much total energy would have been liberated in its gravitational collapse? How long does it take?

Original radius $R_i >> R_f$:

$$\Delta E_g = -\left(E_f - E_i\right) \simeq -E_f \simeq \frac{3}{10}\frac{GM_\odot^2}{R_\odot} \simeq 1.1 \times 10^{41} \text{ J.}$$

Assuming the luminosity is constant throughout the lifetime:

$$t_{\rm KH} = \frac{\Delta E_g}{L_\odot} \sim 10^7 \text{ yr.}$$

This is known as the **Kelvin-Helmholtz timescale**

Too short to account for the age of the Sun, but important during some phases of stellar evolution.

Nuclear Energy

- The real energy source of stars is nuclear energy
- While chemical processes/transitions of electrons in atoms are of the order of 10 eV, nuclear processes involve energies millions of times larger (at the order of MeV)
- A few definitions:
  - Elements are specified by atomic number Z (number of protons)
  - Isotopes of a given element are identified by weight A = Z + N, where number of neutrons N varies
  - E.g., hydrogen has three isotopes, ^{1}H (1 p), ^{2}H = deuterium (1 p + 1 n), ^{3}H = tritium (1 p and 2 n)

Nuclear Energy

- Energy contained in the nucleus is given by Einstein's famous equation

$$E = mc^2$$ Where *m* is the rest mass

- Rest mass of proton, neutron, and electron

$$m_p = 1.67262158 \times 10^{-27}\ \mathrm{kg} = 1.00727646688\ \mathrm{u}$$

$$m_n = 1.67492716 \times 10^{-27}\ \mathrm{kg} = 1.00866491578\ \mathrm{u}$$

$$m_e = 9.10938188 \times 10^{-31}\ \mathrm{kg} = 0.0005485799110\ \mathrm{u}.$$

$1\ \mathrm{u} = 1.66053873 \times 10^{-27}\ \mathrm{kg}$, is the atomic mass unit, 1/12 of the mass of carbon 12

Nuclear Energy

- When protons and neutrons are combined into nucleus, the total mass is slightly less. The difference accounts for the nuclear *binding energy*
- The binding energy is the energy required to split the nucleus into its constituent parts
- When lighter atoms combine into larger atoms, sometimes their combined mass gets smaller, and releases energy, known as *atomic fusion*
- The total mass of four ^1H atoms is 4.031280 u
- After fusion into helium, the ^4_2He atom is 4.002603 u, 0.7% smaller in mass
- Difference in mass dm = 0.028677 u, which corresponds to a energy release $dE = dmc^2 = 26.71$ MeV

Nuclear Energy

- How long (in years) will the Sun last if it shines at its current rate and converts 10% of its mass from hydrogen to helium in nuclear fusion?

Msun = 1.99×10^{30} kg; c = 2.998×10^{8} m/s; 1 yr = 3.15576×10^{7} s

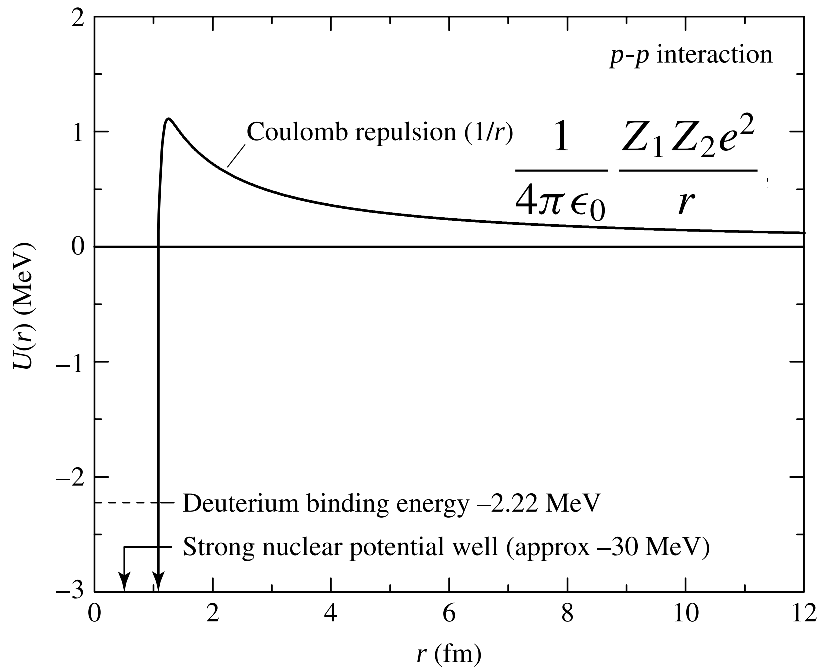
$$E_{\text{nuclear}} = 0.1 \times 0.007 \times M_{\odot} c^2 = 1.3 \times 10^{44}\ \text{J}.$$

$$t_{\text{nuclear}} = \frac{E_{\text{nuclear}}}{L_{\odot}} \sim 10^{10}\ \text{yr},$$

Nuclear timescale. Enough to account for the age of Moon rocks

How does nuclear fusion occur?

- The lighter nucleus have to get (very) close enough to become a heavier nucleus
- Say two hydrogen nucleus wants to collide and form a deuterium nucleus
- Both of the protons have *positive* charges
- They have to overcome the strong *Coulomb barrier* –electric potential energy before it reaches the region where the *strong nuclear force* is in charge. The potential energy goes as ~1/r
- If the energy required comes from the thermal energy of the gas



$$\frac{1}{2}\mu_m\overline{v^2} = \frac{3}{2}kT_{\text{classical}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$$

$$T_{\text{classical}} = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 k r}$$

$\sim 10^{10}$ K

r ~ 1 fm = 10^{-15} m

Femtometer

Temperature at the core of the Sun ~ 1.6 x 10^7 K

Way smaller!

Quantum Mechanical Tunneling

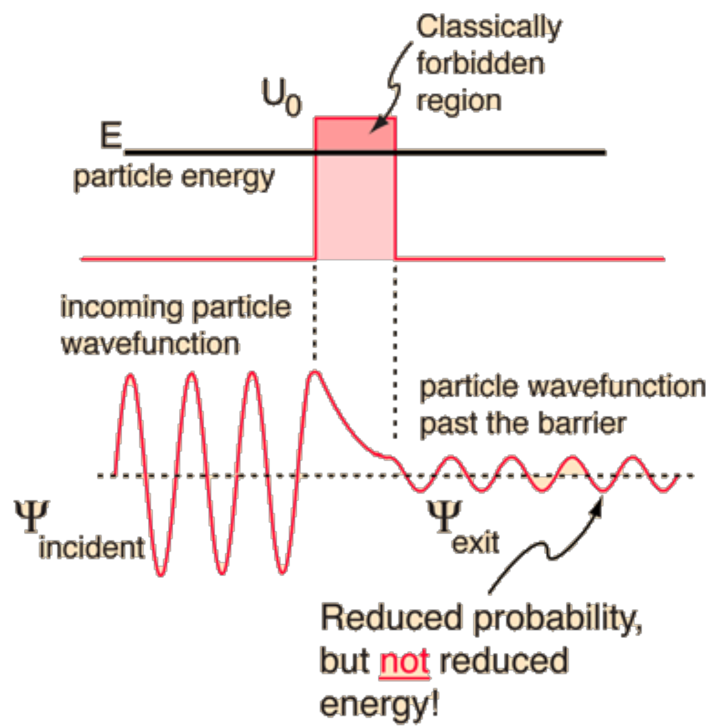
- Quantum physics is the savior!
- Heisenberg's uncertainty principle says the position of a particle is uncertain

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}$$

- de Broglie says matter particles can also display wave properties

$$\lambda = h/p$$

- Our proton can be found *inside* the Coulomb barrier even if the kinetic energy is insufficient, just with some probability — *quantum mechanical tunneling*



Nuclear Fusion: Quantum Effects

Assume that a proton can be within approximately one *de Broglie wavelength* of its target in order to tunnel through the barrier:

$$\lambda = h/p,$$

Rewrite the kinetic energy in terms of momentum

$$\frac{1}{2}\mu_m v^2 = \frac{p^2}{2\mu_m},$$

Setting the distance of closest approach equal to one de Broglie wavelength (so it can tunnel through)

$$\frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{\lambda} = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m}$$

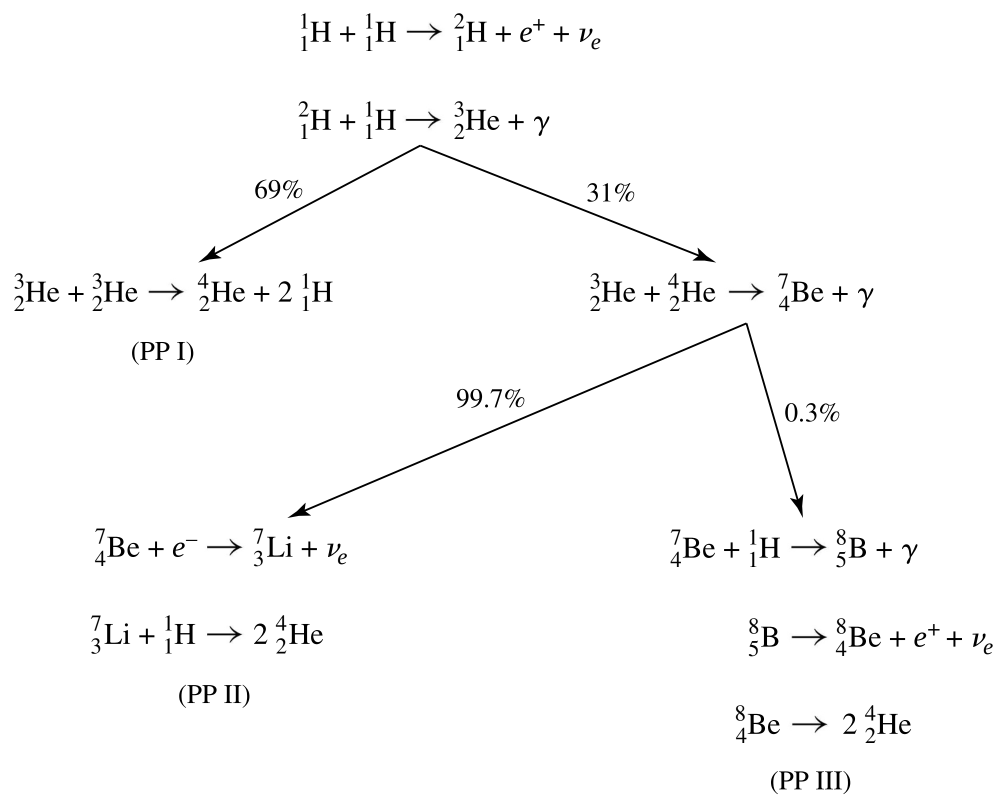
Solving for λ, and substituting r = λ in our solution for classical temperature

$$T_{\text{quantum}} = \frac{Z_1^2 Z_2^2 e^4 \mu_m}{12\pi^2\epsilon_0^2 h^2 k}.$$

~10^7 K. Good enough for the core temperature of the Sun!

Nuclear Fusion Reactions

- Fusion of hydrogen to helium is accomplished not in one step, but through a chain of reactions
- Two main "pathways" for hydrogen burning:
  1. The proton-proton chains (p-p chains). Important for our Sun
  2. The CNO cycle, which requires the presence of carbon, nitrogen, and oxygen, but does not consume them. Important for more massive stars.



The proton-proton chains

Binding energy per nucleon

- To understand the energy release in nuclear reactions, it is useful to consider the *binding energy per nucleon*

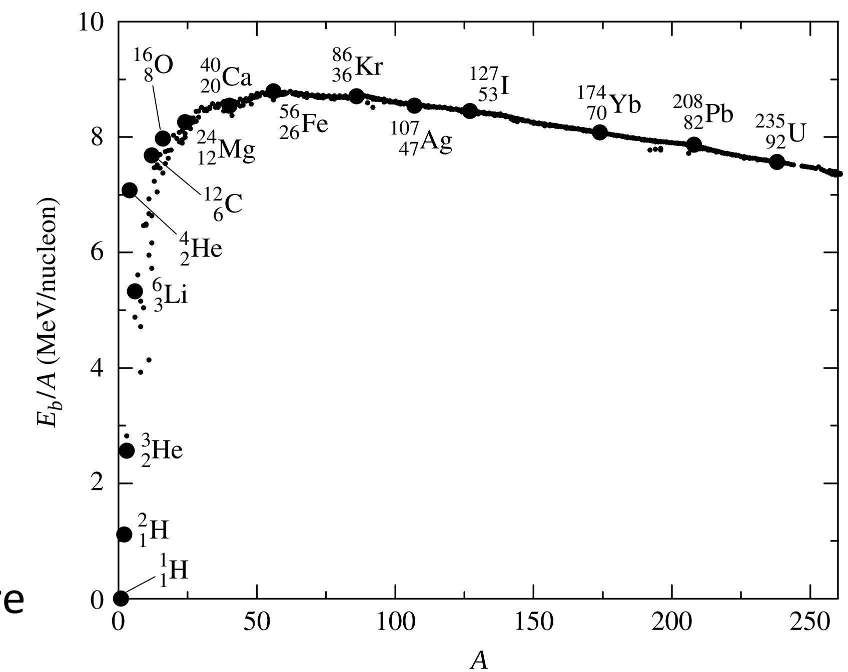
$$E_b/A,$$

$$E_b = \Delta m c^2$$

Mass of the nucleus

$$= \left[Z m_p + (A - Z) m_n - m_{\text{nucleus}} \right] c^2$$

Total mass of individual nucleons



- The more the *binding energy per nucleon*, the lighter the *average mass per nucleon*, the more *stable* the nucleon
- The curve peaks at Fe (Z=26)
- If we combine smaller nuclei into larger ones (nuclear fusion), we will only gain energy up to Fe -> nuclear fusion stops up to Fe

Energy Generation

- Energy generation in stars can be parameterized by an *energy generation rate* ϵ , which is the total energy release per kilogram of material per second (J s^{-1} kg^{-1})

Contribution to the total luminosity from a small mass dm is

$$dL = \epsilon \, dm.$$

For a spherically symmetric star, mass of a thin shell of thickness dr is

$$dm = dM_r = \rho \, dV = 4\pi r^2 \rho \, dr$$

So the interior luminosity as a function of r is

Luminosity gradient equation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon,$$

Energy generation rates in stellar nucleosynthesis

- The reaction rates from different nuclear reactions depend on the **cross-section**
- Usually heavily dependent on **temperature**

Hydrogen Burning

Correction factors ~1

Proton-Proton chains

$$\epsilon_{pp} \simeq \epsilon'_{0,pp}\rho X^2 f_{pp}\psi_{pp}C_{pp}T_6^4$$

Temperature in 10^6 K

with $\epsilon'_{0,pp} = 1.08 \times 10^{-12}\ \mathrm{W\ m^3\ kg^{-2}}$

CNO cycle

$$\epsilon_{\mathrm{CNO}} \simeq \epsilon'_{0,\mathrm{CNO}}\rho X X_{\mathrm{CNO}} T_6^{19.9}$$

with $\epsilon'_{0,\mathrm{CNO}} = 8.24 \times 10^{-31}\ \mathrm{W\ m^3\ kg^{-2}}$

Helium burning

Triple alpha process

$$\epsilon_{3\alpha} \simeq \epsilon'_{0,3\alpha}\rho^2 Y^3 f_{3\alpha} T_8^{41.0}$$

Temperature in 10^8 K

$${}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightleftharpoons {}^8_4\mathrm{Be}$$

$${}^8_4\mathrm{Be} + {}^4_2\mathrm{He} \rightarrow {}^{12}_6\mathrm{C} + \gamma$$

Very strong temperature dependence! E.g., a 10% increase in temperature raises the energy rate by more than x 50!

Outstanding questions

- What supports the stars (from collapsing)?
- What powers the stars?
- What determines the internal structure of the stars?

Energy transport mechanisms

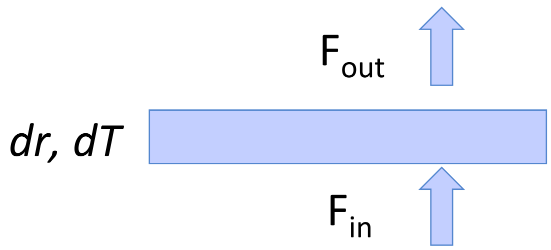
- **Radiation** allows energy produced by nuclear reactions and gravitation to be carried to the surface via **photons**
- **Convection**, with hot, buoyant mass elements carrying excess energy outward while cool elements fall inward
- **Conduction** transports heat via collisions between particles

Energy transport: Radiation

Radiation Flux $F = \sigma T^4$

Flux through a thin spherical shell of thickness *dr* with temperature difference *dT*

$$dF = \sigma T^3 dT$$



Temperature drops as we go out from the center because the gas absorbs energy. The absorption is:

$$dF = -\kappa(r)\rho(r)F(r)dr$$

Combining the two equations above, and using $F(r) = L(r)/4\pi r^2$, we have

$$L(r) = -[16\pi r^2 \sigma T^3 / \kappa(r)\rho(r)]dT/dr$$

More careful analysis results in an additional factor of 4/3, so temperature gradient

$$dT/dr = -\frac{-3\kappa(r)\rho(r)}{4acT^3}\frac{L(r)}{4\pi r^2}$$

with radiation constant $a = 4\sigma / c$

Energy transport: Radiation

$$dT/dr = -\frac{-3\kappa(r)\rho(r)}{4acT^3}\frac{L(r)}{4\pi r^2}$$

- Required temperature gradient *dT/dr* for transporting all energy via radiation becomes **steeper** (more negative) as **flux, density, opacity increase**, or **temperature decreases**
- Under some circumstances *dT/dr* becomes too steep for energy transport by radiation -> **convection** kicks in

Energy transport: Convection

- Initially a Bubble is rising due to buoyancy because the bubble is less dense than its surroundings

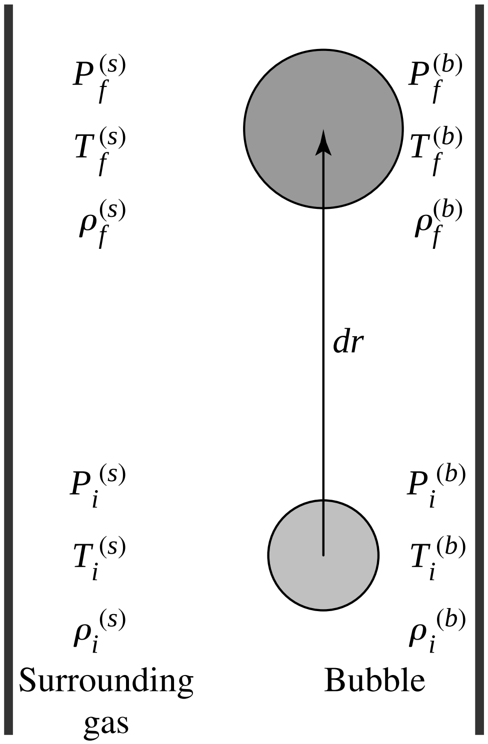
$$\rho_i^{(b)} < \rho_i^{(s)}$$

- If after a infinitesimal distance dr, the bubble now has a greater density than its surroundings

$\rho_f^{(b)} > \rho_f^{(s)}$ ⇨ It will sink again ⇨ No convection

- If the bubble is still less dense than its surroundings

$\rho_f^{(b)} < \rho_f^{(s)}$ ⇨ It will still rise ⇨ Convection



The condition is $$\left|\frac{dT}{dr}\right|_{\text{act}} > \left|\frac{dT}{dr}\right|_{\text{ad}}$$

Actual temperature gradient   Adiabatic temperature gradient

Energy transport: Convection

If energy is mainly carried out by convection, the established temperature gradient is very close to the **adiabatic temperature gradient**

$$\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$$

Using hydrostatic equilibrium and ideal gas law, it can be re-written as

$$\boxed{\left.\frac{dT}{dr}\right|_{\mathrm{ad}} = -\left(1 - \frac{1}{\gamma}\right) \frac{\mu m_H}{k} \frac{G M_r}{r^2}.}$$

Radiation or Convection?

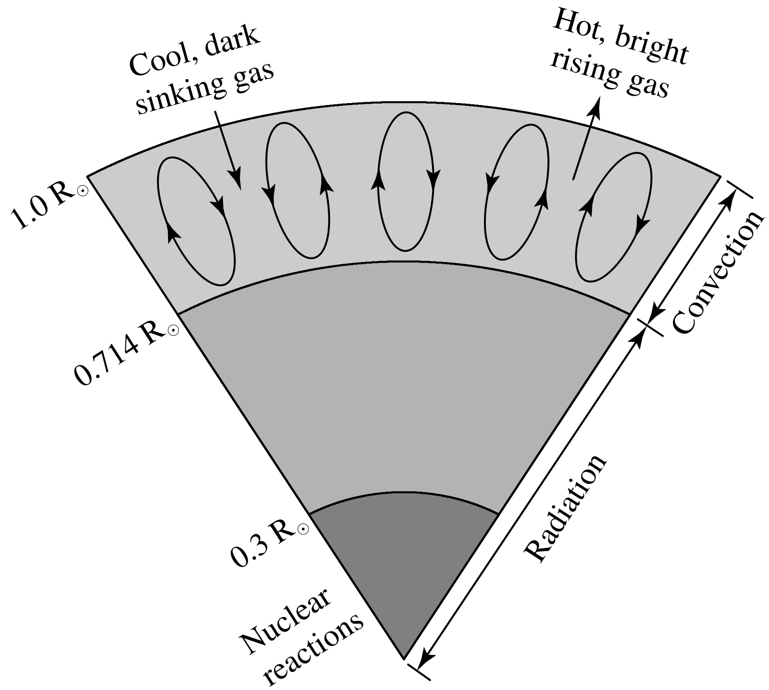
Temperature gradient from radiation

$$dT/dr = -\frac{-3\kappa(r)\rho(r)}{4acT^3}\frac{L(r)}{4\pi r^2}$$

Temperature gradient from convection

$$\left.\frac{dT}{dr}\right|_{ad} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}.$$

- Temperature gradient steeper than adiabatic value: convection occurs
- Shallow temperature gradient: radiation dominates



Schematic diagram of the solar interior

Computer models of stellar interiors

Set of equations

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2}$$ Hydrostatic equilibrium

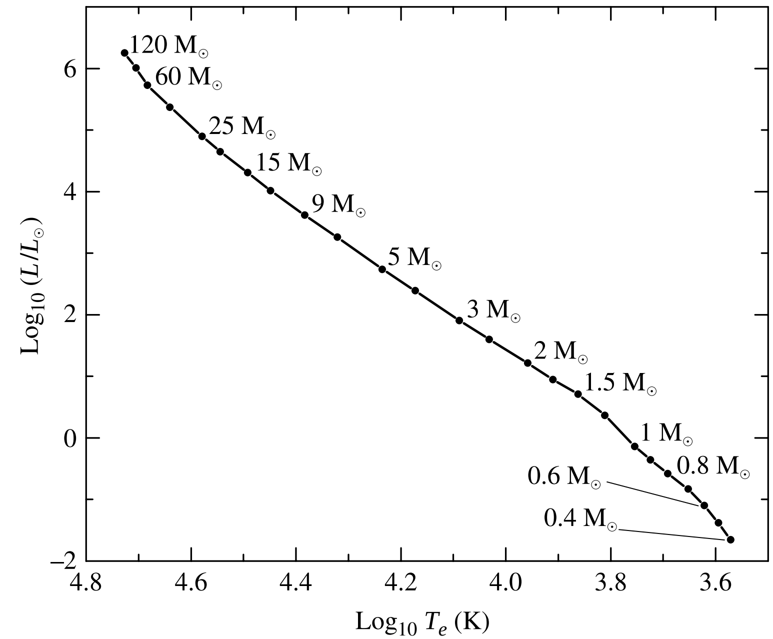
$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$ Mass conservation

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$ Luminosity gradient equation

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\overline{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \quad \text{(radiation)}$$

$$= -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} \quad \text{(adiabatic convection)}$$

$$P_t = \frac{\rho k T}{\mu m_H} + \frac{1}{3}aT^4$$ Ideal gas law and radiation pressure



Models of stars with H burning cores lie on the main sequence!