

Radiative Transfer and Radiation Hydrodynamics

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Abstract. Radiation plays an important role, firstly in determining the structure of stars through the dominant role radiation plays in the energy balance (in some objects also in the momentum balance), secondly because we diagnose astrophysical plasma through the emitted electromagnetic radiation.

These lectures discuss the diagnostic use of optically thick spectral lines and continua with special emphasis on the Sun. Modern methods to solve the equations of radiative transfer and statistical equilibrium are outlined. We stress the importance of solving self-consistently for the coupling between radiation and hydrodynamics for understanding the dynamic outer atmosphere of the Sun and review some results from such radiation hydrodynamics simulations.

1 Introduction

To infer the physical conditions of stars and other astrophysical objects we are almost exclusively restricted to an analysis of the photons that escape from the medium. Before the photons reach us they have interacted with the matter so that the information about local conditions in the atmosphere has been convolved both in space and in time. The diagnostic deciphering of the information content in the radiation is made complicated by the fact that most plasmas both emit and absorb photons at the same time. In addition, photons can travel large distances thus coupling distant parts of the plasma. It is therefore necessary to study the processes of emission, absorption and radiative transfer in order to make proper inferences from observations of radiation.

Radiation also plays an important role in determining the structure of stars through the role of radiation in the energy balance and for some objects also in the momentum balance. Velocity fields in the stellar atmosphere Doppler shift the narrow absorption profiles of spectral lines thus affecting the coupling between the radiation and the local thermodynamic properties of the plasma. Radiative losses damp the temperature increase in the compression phase of a hydrodynamic disturbance. There is thus a strong coupling between the hydrodynamics and the radiation in a dynamic atmosphere and we have to treat this coupling self-consistently to gain an understanding of dynamic phenomena in stellar atmospheres.

The outline of these lectures is as follows: In Section 2 we discuss basic concepts and the basic equation of radiative transfer, in Section 3 we discuss

the concept of height of formation in the context of the diagnostic use of spectral information. In the absence of Local Thermodynamic Equilibrium (LTE) the condition of statistical equilibrium can be used to determine the population densities. These equations are set up in Section 4, a general method of solving non-linear equations is outlined in Section 5 and the particular application to the equations of statistical equilibrium and radiative transfer is given in Section 6. Finally, the equations of radiative transfer and population rate equations are combined with the hydrodynamic equations into the equations of radiation hydrodynamics in Section 7 with special emphasis on recent results for the Solar chromosphere.

2 Equation of Radiative Transfer

We assume a general familiarity with basic radiative transfer theory and will just give an incomplete summary in order to define notation and highlight important aspects. For an excellent and more complete discussion we refer to Mihalas (1978). Furthermore, we restrict the discussion to the case of a one-dimensional plane-parallel atmosphere (all quantities are constant on a plane at a given height in the atmosphere).

Along a ray we look at how the monochromatic specific intensity $I_\nu(s)$ (normally only called the intensity) changes:

$$dI_\nu(s) = [\eta_\nu(s) - \chi_\nu(s)I_\nu(s)] ds \quad (1)$$

where $\eta_\nu(s)$ is called the emissivity, $\chi_\nu(s)$ the opacity (sometimes called extinction or absorption) and ds is the geometrical distance along the ray. Note that the number of photons removed from the ray is proportional to the opacity but also to the intensity itself. The direction dependence enters mainly through the dependence of ds on direction for a given height difference. Both η_ν and χ_ν are often direction *independent* in the absence of velocity fields. However, the opacity can be very direction dependent in the presence of velocity fields.

If we introduce the geometrical height, z , along the normal increasing outwards, the angle between the ray and the normal of the atmosphere, θ , and the directional cosine, $\mu = \cos \theta$, we get one standard form of the equation of radiative transfer:

$$\mu \frac{dI_{\nu\mu}(z)}{dz} = \eta_\nu(z) - \chi_\nu(z)I_{\nu\mu}(z) \quad (2)$$

The right hand side give the source (η_ν) and sink ($\chi_\nu I_{\nu\mu}$) of photons. Since all quantities depend on height the positional coordinate (z) is implicit and normally dropped from the equations.

The emissivity tends to be roughly proportional to the opacity and it is therefore useful to use the ratio of the two instead of the emissivity itself. This ratio is called the source function, $S_\nu \equiv \eta_\nu/\chi_\nu$. Furthermore, it is useful to use

an “optical distance” instead of the geometrical distance. For the radiative transfer a long geometrical distance with small opacity is equivalent to a short geometrical distance with large opacity. The useful quantity is therefore the product of the two. We define the optical depth, τ_ν , from

$$d\tau_\nu = -\chi_\nu dz \quad (3)$$

Note that the optical depth increases downwards along the normal, the geometrical height has opposite direction and we therefore get a minus sign in the equation above.

With these definitions we get the one-dimensional equation of radiative transfer in its standard form:

$$\mu \frac{dI_{\nu\mu}}{d\tau_\nu} = I_{\nu\mu} - S_\nu \quad (4)$$

Note that the source term (S_ν) is now negative and the sink term ($I_{\nu\mu}$) is positive. This is because the optical depth scale has a direction opposite of the normal. For a ray directed towards us (out of the atmosphere) we get a positive value of μ but a negative $d\tau_\nu$.

Since this is a first order ordinary differential equation we need one boundary condition. Normally we know, or can approximate, the intensity incident on each boundary. This means that the boundary condition is given at one boundary for all positive values of μ and at the other boundary for negative values of μ . This fact complicates the solution of the equation. If the atmosphere has an upper boundary but the lower boundary is the interior of the sun we call this a semi-infinite atmosphere and normally use the boundary conditions $I_\nu^-(0) = 0$, $I_\nu^+(\tau_{\max}) = S_\nu(\tau_{\max})$ where superscripts denote the sign of μ . We then get the following solution:

$$I_{\nu\mu}(\tau_\nu) = \begin{cases} \frac{1}{\mu} \int_{\tau_\nu}^{\infty} S_\nu(t) e^{-(t-\tau_\nu)/\mu} dt & \text{if } \mu > 0 \\ \frac{1}{-\mu} \int_0^{\tau_\nu} S_\nu(t) e^{-(\tau_\nu-t)/(-\mu)} dt & \text{if } \mu < 0 \end{cases} \quad (5)$$

Note the use of $(-\mu)$ for $\mu < 0$ to get a positive quantity.

The interpretation of the solution is that we get a contribution to the outgoing intensity from a given depth, t , over a given small distance along the ray, dt/μ (where the factor $1/\mu$ accounts for the projection of the distance dt along the normal to the distance along the ray) which is equal to the source function there exponentially attenuated with the optical distance from t to the point where we are evaluating the intensity, τ_ν . This optical distance equals the optical distance along the normal $(t - \tau_\nu)$ projected to the direction of the ray (the factor $1/\mu$ in the exponential). These contributions are summed up (the integral) from all depths below the evaluation point. The interpretation of the solution for the incoming radiation is equivalent.

This solution is called the *formal solution* because the source function has to be known (in addition to the opacities giving the optical depth scale). In

the solar outer atmosphere the source function depends on the intensity itself such that a direct integration as in (5) is not possible. We then have to use an iterative method to find the solution. In all such iterative methods we need to evaluate the formal solution in each iterative step. It is thus important to find efficient methods for the formal solution even when we do *not* know the source function a priori.

One standard method for the evaluation of the formal solution is the Feautrier method (Feautrier 1964). The method is efficient, stable, accurate and gives the intensity at all points along the ray in one step. For more accurate versions of the Feautrier method see Kunasz & Hummer (1974), Auer (1976) and Rybicki & Hummer (1991).

The opacity can be split into an absorption part and a scattering part. Absorption can consist of excitation of an atom by absorption of a photon followed by a collisional de-excitation. The net effect is then the destruction of a photon converting the energy to local thermal energy. If the de-excitation instead is radiative we end up with the photon just changing direction and no coupling to the local conditions. Such a process is called a scattering event. The distinction between absorption and scattering becomes more problematic if atoms with more than two levels are considered but the two-level case is still instructive. Using this distinction and assuming isotropic scattering we get an emissivity consisting of a thermal part and a scattering part:

$$\eta_\nu = \kappa_\nu B_\nu + \sigma_\nu J_\nu \quad (6)$$

where κ_ν is absorption, σ_ν scattering, $\chi_\nu = \kappa_\nu + \sigma_\nu$, B_ν the Planck function and J_ν the mean intensity:

$$J_\nu(z) = \frac{1}{2} \int_{-1}^1 I_{\nu\mu}(z) d\mu \quad (7)$$

The source function can then be written as

$$S_\nu = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu \quad (8)$$

It is the scattering part that introduces a global coupling in the source function through the dependency on the mean intensity.

3 Height of Formation

3.1 Contribution Functions

For the diagnostic use it is of interest to find the depth of formation of the observed quantity. From equation (5) it is obvious that the intensity is formed over a range of depths. It is useful to define a *contribution function to the intensity*, C_I , with the property:

$$I_{\nu\mu}(0) = \int C_I(x) dx \quad (9)$$

Note that two quantities enter into the definition of the contribution function; the contribution *to* a given observable quantity (in this case the intensity) *on* a given depth-scale (here denoted with x). The integration is performed over the whole atmosphere. Since many different integrands give the same integrated quantity we need additional constraints in our definition, see Magain (1986) for a discussion. Using the integrand from the formal solution is an appropriate choice. If we use monochromatic optical depth as the depth variable we get from equation (5):

$$C_I(\tau_\nu) = \frac{1}{\mu} S_\nu(\tau_\nu) e^{-\tau_\nu/\mu} \quad (10)$$

The monochromatic optical depth-scale is not very useful when comparing the formation depth at different wavelengths. To transform the contribution function to another depth-scale we use:

$$C_I(x) dx = C_I(\tau_\nu) d\tau_\nu \quad (11)$$

Some useful contribution functions are the contribution function to the intensity on a geometric height-scale (z), a standard optical depth-scale (τ_0 , where χ_0 is the opacity at a standard wavelength, usually 500 nm) and the logarithm of the standard optical depth ($\lg \tau_0$):

$$C_I(z) = \frac{1}{\mu} S_\nu(\tau_\nu) e^{-\tau_\nu/\mu} \chi_\nu \quad (12)$$

$$(13)$$

$$C_I(\lg \tau_0) = \frac{\ln 10}{\mu} S_\nu(\tau_\nu) e^{-\tau_\nu/\mu} \frac{\chi_\nu}{\chi_0} \tau_0 \quad (14)$$

Note that these contribution functions give information on where the *intensity* is formed. If we are interested in where the *absorption* is formed we *cannot use the contribution function to the intensity* but need to consider the contribution function to the *relative absorption* defined as:

$$R_\nu \equiv \frac{I_c - I_{\nu\mu}}{I_c} \quad (15)$$

where I_c is the continuum intensity. Note that R_ν is zero for the continuum and is negative for an emission line.

To deduce the contribution function for the relative absorption we need to write the transfer equation for the relative absorption, find the formal solution and identify the integrand as the contribution function; all steps analogous to how the contribution function to the intensity was found. See Magain (1986) for details.

The relative absorption is typically formed higher than the intensity. In extreme cases the absorption may be formed at a very different place than the intensity (*e.g.* telluric absorption lines where the intensity is formed in the Sun and the absorption in the Earth's atmosphere).

Typical contribution functions to the intensity on a geometric height-scale (note the specification of both the quantity (intensity) and the depth-scale) are shown in Fig. 1.

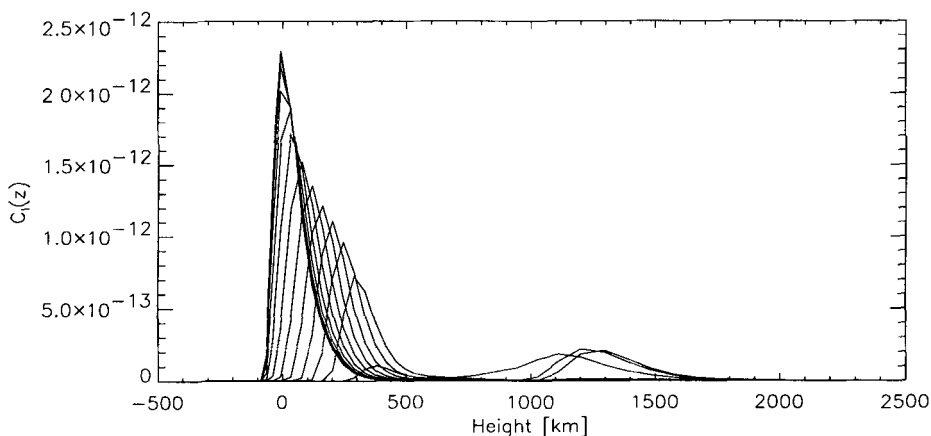


Fig. 1. Typical contribution functions to the intensity on a height scale at different frequencies from continuum (lowest height) to the line center for a Solar atmosphere. Note a typical width of about 200 km.

3.2 Response Functions

We are often interested in deducing the bulk velocity from the Doppler-shift of a line or the temperature change from the change of intensity. In these cases we should use the *response function* instead of a contribution function. The response function gives how a given observed quantity changes (*e.g.* Doppler-shift of line-center) when we change a physical parameter in the atmosphere (*e.g.* velocity) as a function of depth. There are thus three defining quantities in a response function. From this definition we get:

$$\Delta V = \int R_{\Delta V, v}(x) v(x) dx \quad (16)$$

where ΔV is the Doppler-shift of line-center and $v(x)$ is the vertical velocity. The Doppler-shift response function to bulk-velocity is often similar to the contribution function to the relative absorption. The width of the response function makes it impossible to measure velocity variations that have shorter typical length-scales. Specifically, it is not possible to detect high frequency waves (with a wavelength shorter than the width of the response function) in a stellar atmosphere. Waves with a wavelength not much longer than the width of the response function will give Doppler-shift amplitudes that are substantially smaller than the velocity amplitude of the wave. This is illustrated in Fig. 2.

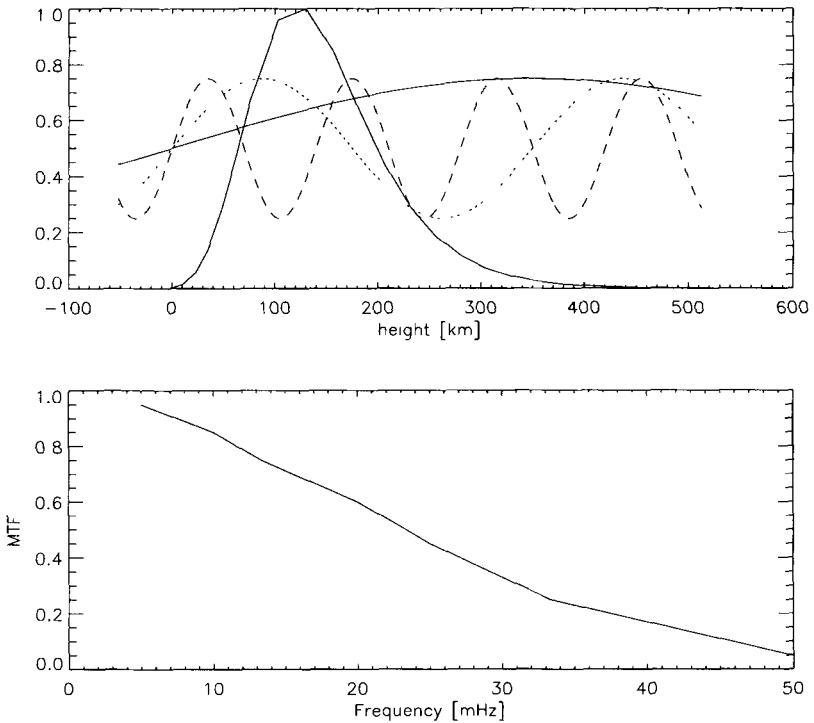


Fig. 2. *Upper panel:* Response function to line-center Doppler-shift for a typical spectral line in the solar atmosphere (*solid*) and three sinusoidal velocity fields with frequency 5 mHz (*solid*), 20 mHz (*dotted*) and 50 mHz (*dashed*). *Lower panel:* Amplitude of Doppler-shift compared to amplitude of the sinusoidal wave as function of frequency. Waves with a frequency of 23 mHz give a Doppler-shift of only half the amplitude of the wave due to the width of the response function.

4 Statistical Equilibrium

We here give a slightly simplified, phenomenological, picture of what is involved in radiative transfer problems in order to give a feel for what it is all about. For a more rigorous treatment the reader is again referred to standard textbooks like Mihalas (1978).

In order to calculate the specific intensity from the formal solution (Equation 5) we need to know the source function, S_ν and the monochromatic optical depth, τ_ν . The latter can be calculated if we know the monochromatic opacity, χ_ν . The opacity for a given process is equal to the cross-section for the process at a given frequency multiplied by the number density of the particle. The total opacity at a given frequency is the sum of the opacities of the processes that can take place at that frequency. To calculate the opacity we thus need cross-sections which, at least in principle, are available from atomic physics and we need population densities.

If we assume Local Thermodynamic Equilibrium (LTE) the population density of a given energy level of an atom is given by the Saha and Boltzmann equations. The ratio of the population density to the mass density is then given by the abundance of the element and the *local* values of the electron density and the temperature. With a given temperature structure as a function of height we can calculate the mass density and the electron density from the hydrostatic equilibrium and charge conservation equations (see *e.g.*, Gray 1992). If we in addition assume that the source function is given by the Planck function we can calculate all we need for the formal solution. Radiative transfer under the assumption of LTE is thus trivial in principle. In practice the calculation of a spectrum from a stellar atmosphere with given temperature structure is still non-trivial due to the need for atomic data (line strengths, broadening parameters) for millions of lines.

If we cannot assume LTE the problem becomes much more complex. The source functions will no longer be given by the Planck function and the population densities will not be given by the Saha and Boltzmann equations. In a *static* atmosphere we get the necessary extra equations from demanding that the population density of a given energy level shall be constant in time. We end up with the equations of statistical equilibrium:

$$n_i \sum_{j \neq i}^{n_i} P_{ij} - \sum_{j \neq i}^{n_i} n_j P_{ji} = 0 \quad (17)$$

where n_i is the population density of energy level i , P_{ij} is the transition probability of a transition from energy level i to energy level j and n_i is the number of energy levels considered. The first term thus gives the number of transitions per unit time from energy level i to all other energy levels and the second term gives the number of transitions into energy level i . Statistical equilibrium requires these rates to be equal. We have one such equation for

each energy level considered. This set is linearly dependent and we have to replace one of the equations with the condition of particle conservation,

$$\sum_{j=1}^{n_i} n_j = n_{tot} \quad (18)$$

where n_{tot} is the total number of particles of that element.

To solve this coupled set of equations we only need to know the transition probabilities since the total number of particles of a given element is given by the abundance and the mass density. The problem lies in the fact that the transition probabilities in general depend on the radiation field. We have

$$P_{ij} = R_{ij} + C_{ij} \quad (19)$$

with C_{ij} being the collisional transition probability (in principle known from atomic physics) and R_{ij} the radiative transition probability. For transitions between lower bound state i and upper bound state j we have:

$$\begin{aligned} R_{ji} &= A_{ji} + B_{ji} \bar{J}_{ij} \\ R_{ij} &= B_{ij} \bar{J}_{ij} \end{aligned} \quad (20)$$

with A_{ji} , B_{ij} and B_{ji} the Einstein coefficients for spontaneous emission, absorption and stimulated emission, respectively. All these are given by atomic physics. \bar{J}_{ij} is the absorption profile ($\phi_{\nu\mu}$) weighted integrated mean intensity:

$$\bar{J}_{ij} = \frac{1}{2} \int_{-1}^1 \int_0^{\infty} \phi_{\nu\mu} I_{\nu\mu} d\nu d\mu \quad (21)$$

For transitions between two bound states the opacity and emissivity are

$$\chi_{\nu\mu} = \lambda_{\nu c} + \frac{h\nu_{ij}}{4\pi} B_{ij} \phi_{\nu\mu} (n_i - \frac{g_i}{g_j} n_j) \quad (22)$$

$$\eta_{\nu\mu} = \eta_{\nu c} + \frac{h\nu_{ij}}{4\pi} \phi_{\nu\mu} A_{ji} n_j \quad (23)$$

where subscript c indicates background processes that do not depend on the population densities. g_i and g_j are the statistical weights for the lower and upper levels, respectively. The line opacity is related to the upward radiative rate. The last term in the line opacity comes from the treatment of stimulated emission as negative absorption.

The radiative rates introduce a *global* coupling since the intensity depends on conditions throughout the atmosphere, see Equation 5. In order to find the population densities we have to know the intensities and in order to find the intensities we need the population densities. The equations of statistical equilibrium and radiative transfer constitute a coupled set of non-local, non-linear equations that have to be solved simultaneously. A strategy for

solving sets of non-linear equations is by linearization. The general technique is outlined in Section 5 with the particular application to the coupled set of statistical equilibrium and radiative transfer equations given in Section 6.

5 Linearization

A classical method to solve a non-linear equation or a set of non-linear equations is the Newton-Raphson method. We may formulate this iterative method as a perturbation method and will illustrate the procedure by first applying it to a simple non-linear equation:

$$x^2 = 2 \quad (24)$$

At the n 'th stage in an iterative method we do not have the exact solution x but instead an approximation to the solution, $x^{(n)}$. When this approximation is inserted in the equation we do not get the right hand side exactly but have an error, $E^{(n)}$:

$$(x^{(n)})^2 = 2 + E^{(n)} \quad (25)$$

We look for an addition to the current approximation that will fulfill the equation:

$$(x^{(n)} + \delta x)^2 = 2 \quad (26)$$

Expanding the parenthesis we get

$$(x^{(n)})^2 + 2x^{(n)}\delta x + \delta x^2 = 2 \quad (27)$$

The essence of linearization consists of neglecting non-linear terms in the perturbation (the term δx^2) and defining an approximate correction, $\delta x^{(n)}$, from the resulting equation:

$$(x^{(n)})^2 + 2x^{(n)}\delta x^{(n)} = 2 \quad (28)$$

Subtracting Equation 25 results in an equation for the approximate correction:

$$2x^{(n)}\delta x^{(n)} = -E^{(n)} \quad (29)$$

The iterative procedure thus starts by calculating the error from the current estimate of the solution (Equation 25), continues by calculating the approximate correction (Equation 29) and adding that correction to get the next estimate of the solution:

$$E^{(n)} = (x^{(n)})^2 - 2 \quad (30)$$

$$\delta x^{(n)} = -\frac{E^{(n)}}{2x^{(n)}} \quad (31)$$

$$x^{(n+1)} = x^{(n)} + \delta x^{(n)} \quad (32)$$

To start the iteration we need a starting approximation. Starting with $x^{(1)} = 1$ we get the following values:

n	$x^{(n)}$	$\lg x - \sqrt{2} $
1	1	-0.4
2	1.5	-1.1
3	1.416	-2.6
4	1.414216	-5.7
5	1.414213562	-11.8

The convergence is rapid with about double the number of significant figures per iteration — the convergence is quadratic. One may also note that the equation has two solutions but we get convergence only to one; which one depends on the starting approximation.

Linearization is a very general and powerful technique. Instead of one equation we may have a system of equations (like the statistical equilibrium equations). We may very easily add more constraints or equations to the system (like an energy equation instead of assuming a given temperature stratification). In the linearization step (defining the approximate correction, Equations 27–28) we may introduce additional approximations (like an approximate treatment of the radiative transfer). However, if additional approximations are introduced we lose the property of quadratic convergence.

The approximations introduced do *not* affect the accuracy of the method, only the convergence rate (or lack thereof). The accuracy is only set by the accuracy with which we can calculate the error from a given approximation. In the next section we will show how these methods can be used to solve the coupled equations of statistical equilibrium (SE) and radiative transfer (RT).

6 Linearization of the SE and RT Equations

Following the steps of the previous section we introduce a current approximation in the statistical equilibrium equations (Equation 17):

$$n_i^{(n)} \sum_{j \neq i}^{n_i} P_{ij}^{(n)} - \sum_{j \neq i}^{n_i} n_j^{(n)} P_{ji}^{(n)} = E_i^{(n)} \quad (33)$$

We perturb the population densities and all variables that depend on the population densities:

$$n_i^{(n+1)} = n_i^{(n)} + \delta n_i^{(n)} \quad (34)$$

$$P_{ij}^{(n+1)} = P_{ij}^{(n)} + \delta P_{ij}^{(n)} \quad (35)$$

and require

$$n_i^{(n+1)} \sum_{j \neq i}^{n_i} P_{ij}^{(n+1)} - \sum_{j \neq i}^{n_i} n_j^{(n+1)} P_{ji}^{(n+1)} = 0 \quad (36)$$

Expansion, linearization and subtraction gives

$$\begin{aligned} \delta n_i^{(n)} \sum_{j \neq i}^{n_i} P_{ij}^{(n)} + n_i^{(n)} \sum_{j \neq i}^{n_i} \delta P_{ij}^{(n)} - \sum_{j \neq i}^{n_i} \delta n_j^{(n)} P_{ji}^{(n)} \\ - \sum_{j \neq i}^{n_i} n_j^{(n)} \delta P_{ji}^{(n)} = -E_i^{(n)} \end{aligned} \quad (37)$$

with the perturbed rates given by Einstein coefficients and perturbed intensities. For an upward bound-bound rate we get

$$\delta P_{ij}^{(n)} = B_{ij} \delta \bar{J}_{ij}^{(n)} = B_{ij} \frac{1}{2} \int_{-1}^1 \int_0^\infty \phi_{\nu\mu} \delta I_{\nu\mu}^{(n)} d\nu d\mu \quad (38)$$

since the collisional rates and profile functions do not depend on the population densities and we assume the electron density and temperature to be given.

We still have to write the perturbed intensities in terms of perturbations in population densities and it is here we need the equations of radiative transfer.

We therefore take the transfer equation

$$\mu \frac{dI_{\nu\mu}^{(n)}}{dz} = -\chi_{\nu\mu}^{(n)} I_{\nu\mu}^{(n)} + \eta_{\nu\mu}^{(n)} \quad (39)$$

add perturbations, linearize and subtract the transfer equation to obtain

$$\mu \frac{d}{dz} \delta I_{\nu\mu}^{(n)} = -\chi_{\nu\mu}^{(n)} \delta I_{\nu\mu}^{(n)} - I_{\nu\mu}^{(n)} \delta \chi_{\nu\mu}^{(n)} + \delta \eta_{\nu\mu}^{(n)} \quad (40)$$

We introduce an equivalent source function perturbation from

$$\delta S_{\nu\mu}^{(n)} = \delta \eta_{\nu\mu}^{(n)} / \chi_{\nu\mu}^{(n)} - I_{\nu\mu}^{(n)} \delta \chi_{\nu\mu}^{(n)} / \chi_{\nu\mu}^{(n)} \quad (41)$$

and define a monochromatic optical depth along the ray from

$$d\tau_{\nu\mu}^{(n)} = -\chi_{\nu\mu}^{(n)} dz / \mu \quad (42)$$

We can then write a transfer equation for the *perturbations*:

$$\frac{d}{d\tau_{\nu\mu}^{(n)}} \delta I_{\nu\mu}^{(n)} = \delta I_{\nu\mu}^{(n)} - \delta S_{\nu\mu}^{(n)} \quad (43)$$

Through the formal solution of the transfer equation the intensity perturbation can be written as an integral over depth of the equivalent source function perturbation. This perturbation can be expressed in terms of the opacity and emissivity perturbations (using Equations 22–23) and since these depend on the population density perturbations we have achieved our goal of writing everything in terms of population density perturbations. We have a coupled set of equations for the population density corrections; one equation for each atomic energy level for each depth in the atmosphere. We thus have a matrix equation of order $n_l n_\tau$ where n_τ is the number of depth-points. This matrix is in principle full because the intensity perturbation at one depth depends on the population density perturbations at all other depths through the transfer equation. Calculating the matrix is time-consuming because we need to calculate on the order of $(n_l n_\tau)^2$ exponentials in the formal solution. Solving the matrix equation is also time consuming since the coefficient matrix is full.

In order to have a more efficient method we should find additional approximations that contain the basic physics (in order to keep a decent convergence rate) but lead to savings in the time it takes to construct the matrix and also in the solution time.

One such approximation was found by Scharmer (1981) who extended the Eddington-Barbier relation by replacing the formal solution integral with a one-point quadrature formula. The intensity at one point is thus approximated by the source function in another point multiplied by some weight,

$$I_{\nu\mu}(\tau_{\nu\mu}) = \omega S_{\nu\mu}(\tau'_{\nu\mu}) \quad (44)$$

By demanding that the formula shall be exact for a linear source function Scharmer arrived at the following quadrature points and weights:

$$\begin{cases} \tau'_{\nu\mu} = \tau_{\nu\mu} + 1, & \omega = 1, & \mu > 0 \\ \tau'_{\nu\mu} = \frac{\tau_{\nu\mu}}{1 - e^{-\tau_{\nu\mu}}} - 1, & \omega = 1 - e^{-\tau_{\nu\mu}}, & \mu < 0 \end{cases} \quad (45)$$

For the outgoing intensity these relations say that the intensity is equal to the source function one unit of optical depth further down along the ray. This one-point quadrature formula is exact for a linear source function but is also a rather good approximation when the source function is not a linear function of optical depth because of the rather narrow contribution function to the intensity. Since the approximation is only used in the calculation of the approximate correction and *not* in the calculation of the error it will only affect the convergence rate and not the accuracy of the result.

Experience has shown that the Scharmer operator is close to the optimal choice for the convergence rate. For the outgoing intensity we need no exponentials at all and for the incoming intensity we need only $n_l n_\tau$ exponentials thus reducing the work needed to construct the matrix by a large factor. The resulting matrix is also close to a banded structure with large portions identically zero. The quadrature point for the incoming intensity approaches

$\tau_{\nu\mu}/2$ for small values of $\tau_{\nu\mu}$. With seven grid-points per decade of optical depth this corresponds to just two points higher up in the atmosphere. We thus get only two sub-diagonals with non-zero elements below the diagonal of the matrix. This fact leads to large savings in the solution time.

The main draw-back of the Scharmer operator is closely related to the advantages. The operator takes explicitly into account the non-local nature of the radiative transfer and this leads to good convergence properties. However, the generalization to multi-dimensional radiative transfer is almost impossible when the treatment of the radiative transfer is non-local. The methods of choice for multi-dimensional radiative transfer problems are therefore the local methods developed by Olson et al. (1986) and Rybicki & Hummer (1991). See Hubeny (1992) for a review. A schematic illustration of the different radiative transfer operators is shown in Fig. 3

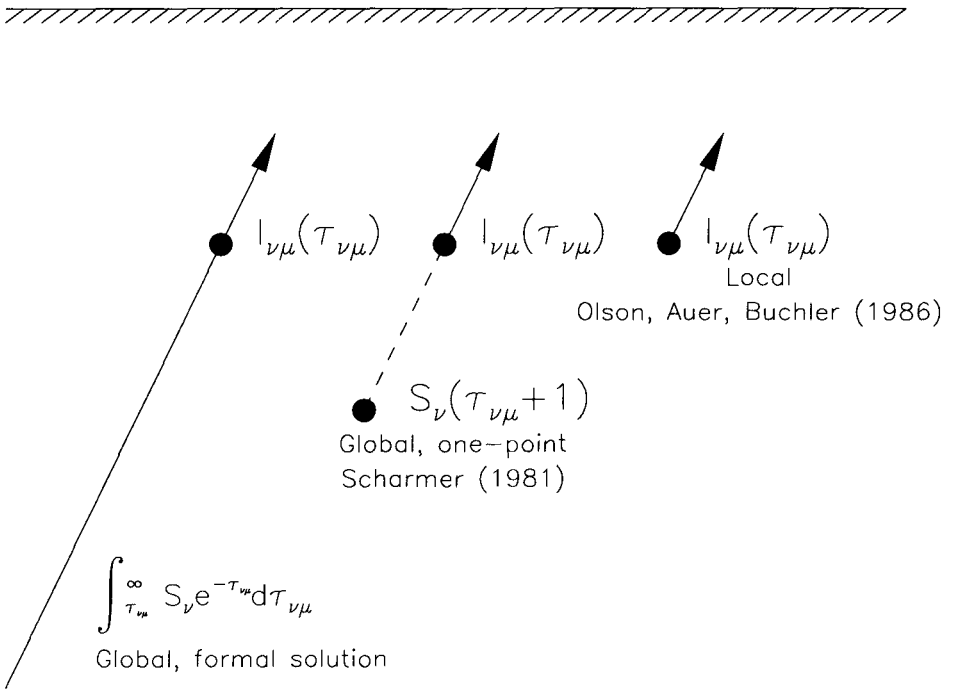


Fig. 3. Schematic illustration of different radiative transfer operators. The formal solution writes the intensity at optical depth $\tau_{\nu\mu}$ along the ray as an integral over depth and is thus global. The Scharmer operator replaces the integral over depth with a one-point quadrature formula. Since the quadrature point is non-local the operator is still global. The OAB operator is completely local.

7 Radiation Hydrodynamics

The previous discussion on how to solve for the outgoing intensity assumed that the atmosphere was given, in particular the variation of electron density and temperature throughout the atmosphere. This is called the forward solution of the *diagnostic* problem. However, the *structure* of the atmosphere, including the bulk velocity, density, electron density and temperature, is affected by the radiation. To solve for these four additional quantities in a one-dimensional plane-parallel atmosphere we need four additional equations. These are the equations of conservation of momentum, mass, charge and energy. The momentum equation may have a significant contribution from radiation pressure but in the solar case this radiative term can be neglected. The energy equation will be dominated by the radiative terms since the energy transport in stellar atmospheres often is dominated by radiation. The other dominant term in the energy equation may be energy transport by convection. In outer parts of the atmosphere we may have large terms in the energy balance from waves and shocks.

For a static atmosphere we neglect all variations with time and set the bulk velocity to zero. The above four equations then reduce to three: hydrostatic equilibrium, conservation of charge and energy. If we in addition assume LTE we have reduced the radiation hydrodynamics problem to the standard “classical” model atmosphere problem where we assume

- one dimensional geometry (either plane parallel or spherical)
- hydrostatic equilibrium (no systematic velocities and no change in time)
- convective-radiative equilibrium
- Local Thermodynamic Equilibrium (LTE)

Convective energy transport is in the classical models described with a mixing-length phenomenological formalism (*e.g.*, Mihalas 1978).

The assumption of LTE means that the distribution of population densities over energy states can be calculated from the Boltzmann and Saha equations using *local* values of the temperature and electron density restricting the effects of the potentially global coupling from the radiation. This is an enormous simplification of the problem. There is often no *a priori* justification for the use of this assumption other than the reduction of computational effort.

The above assumptions lead to only three parameters describing the stellar atmosphere: the effective temperature, the acceleration of gravity and the abundances of the elements. These few parameters all have a clear physical meaning. The fact that such classical stellar atmosphere models can reproduce observed spectra to some detail is one of the major success stories in modern astrophysics. Additional fudge parameters are, however, often needed to cover-up for the inaccuracy of the assumptions potentially masking important aspects of the physics. Such fudge parameters include the so-called microturbulence (giving additional line broadening) and macroturbulence.

In the outer parts of stellar atmospheres it is clear that LTE is a bad approximation. It is possible to replace this assumption with the statistical equilibrium equations and still solve the static model atmosphere problem. See Auer & Mihalas (1969, 1970, 1972), Mihalas & Auer (1970), Werner (1986, 1987, 1989), Anderson (1985, 1989, 1991), Dreizler & Werner (1992, 1993) and Hubeny & Lanz (1995) for examples.

For other trends in stellar atmosphere modelling see Carlsson (1995).

In the next sections we look at some specific aspects of the radiation hydrodynamics problem. We set the scene by deriving the simple condition of radiative equilibrium in Section 7.1, discuss some numerical problems and how they can be overcome in Sections 7.2–7.3, outline how the equations can be solved in Section 7.4 and discuss results in Section 7.5.

7.1 Radiative Equilibrium

If the only mode of energy transport is through radiation we get the condition of radiative equilibrium as our energy equation in a static atmosphere. Integrating the RT equation (2) over angle we get

$$\frac{dF_\nu}{dz} = 2\pi \int_{-1}^1 \chi_{\nu\mu} (S_{\nu\mu} - I_{\nu\mu}) d\mu \quad (46)$$

where F_ν is the monochromatic radiation flux density. Integrating over frequency we get the total flux divergence:

$$\frac{dF}{dz} = 2\pi \int_0^\infty \int_{-1}^1 \chi_{\nu\mu} (S_{\nu\mu} - I_{\nu\mu}) d\mu d\nu = 0 \quad (47)$$

with the last equality being the condition of radiative equilibrium.

This energy equation only specifies that the total radiation flux density does not change with height but does not specify the value of the flux density. We therefore have to specify the flux density at least at one height in addition to using the above equation:

$$F = \sigma T_{\text{eff}}^4 \quad (48)$$

with σ being the Stefan-Boltzmann constant and T_{eff} the effective temperature of the star.

If the opacity and source functions are isotropic (not unreasonable for stationary atmospheres) we can perform the integration over angle in (47):

$$\frac{dF}{dz} = 4\pi \int_0^\infty \chi_\nu (S_\nu - J_\nu) d\nu = 0 \quad (49)$$

7.2 Preconditioning

One problem in implementing the condition of radiative equilibrium in a code is that the driving term is the *difference* between the mean intensity and the source function. At large optical depths these quantities are almost equal and the difference may be lost in the numerical truncation errors. If possible one should then *precondition* the equations by analytically taking out the large terms that almost cancel, see Rybicki (1972) and Scharmer & Carlsson (1985) for examples.

7.3 Treatment of Millions of Spectral Lines

In an ordinary star there are millions of spectral lines such that a standard discretization, adequately sampling the variations in opacity and mean intensity, would require an enormous number of frequency points. Two basic methods exist to overcome this problem.

In **Opacity Sampling (OS)** a large number of points are distributed over the spectrum. Each individual spectral line is not properly sampled but if the number of points is large enough the *statistical* properties are properly sampled and the flux divergence is adequately represented. The calculated *spectrum* is, however, undersampled and can not be compared directly to an observed spectrum. For cool stars on the order of 10^4 frequency points are needed.

In the **Opacity Distribution Function (ODF)** approach the opacities are redistributed within narrow wavelength bins (typically 50 \AA) to create a smoother function of opacity as a function of wavelength. This smoother function can be sampled with fewer frequency points than the original opacity distribution, see Fig. 4. On the order of 500 points may be needed, a factor of 20 fewer than with the opacity sampling method. One disadvantage is that high opacities are lined up in depth and this may be unphysical because different spectral lines may dominate the opacity at different depths, mainly because of differences in temperature and therefore ionization balance. Another disadvantage is that the calculation of the opacity distribution functions is timeconsuming and has to be redone if the abundances are changed.

In LTE one may even sample ODFs over the whole spectrum if the variation of the Planck function is taken into account. In this way one may approximately calculate the radiative flux divergence with only a handful of frequency points. This is the approach chosen to treat the radiative energy transport in the 3D radiation-hydrodynamic simulations by Nordlund & Stein, see Nordlund (1982) for details.

7.4 Complete Linearization

Treating the forward diagnostic problem we ended up with one equation for each energy level considered (see Section 6). The full radiation hydrodynamics problem is not much more complicated from a methodological point of

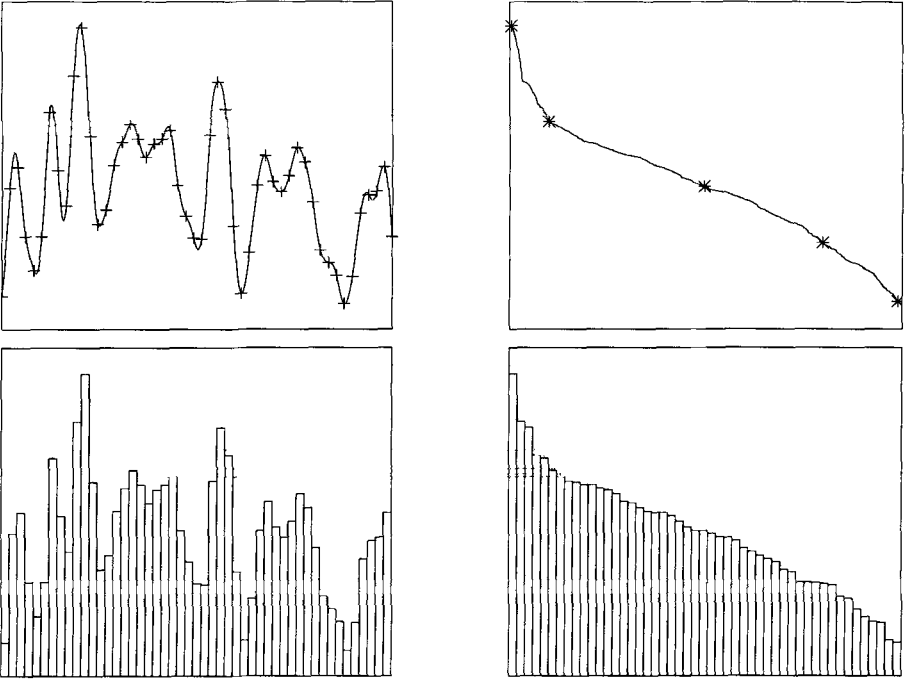


Fig. 4. Opacity distribution function method. *Left panels:* The original opacity as a function of wavelength within a narrow band. *Right panels:* The same opacities but now sorted with all the high opacities at one end of the wavelength band. The lines between the bottom panels show how every second of the ten largest opacities have been moved in the sorting process. Only a handful of points is needed to well approximate the right hand distribution while the original distribution demands a large number of points (*upper panels*).

view. We need four extra equations for the four extra variables bulk velocity, density, electron density and temperature: conservation of momentum, mass, charge and energy. Taking into account the time-variation means the statistical equilibrium equations are replaced by the rate equations. They state that the net rate into an energy level no longer is zero but is equal to the time derivative plus an advection term giving the net inflow into a volume element:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial z}(n_i v) = \sum_{j \neq i}^{n_i} n_j P_{ji} - n_i \sum_{j \neq i}^{n_i} P_{ij} \quad (50)$$

The method of linearization can still be used. The difference in the rate equations is that we get many more terms. The δP_{ij} terms in Equation (37) now get contributions also from δC_{ij} since the collisional rates depend on

both electron density and temperature. We also get terms from the profile functions. From the $\delta\chi_{\nu\mu}$ term we used to have only δn_i and δn_j terms but we now need the whole expression:

$$\delta\chi_{\nu\mu} = \sum_x \frac{\partial\chi_{\nu\mu}}{\partial x} \delta x \quad (51)$$

with x being all the variables we solve for ($\{n_i\}_1^{n_t}, v, \rho, n_e, T$). All perturbations thus have to be expanded into perturbations of the variables we solve for. In the diagnostic problem the radiative transfer equation was incorporated into the statistical equilibrium equations through the use of Scharmer's operator. We got expressions of an equivalent source function perturbation $\delta S_{\nu\mu}$ that could be written in terms of δn_i and δn_j through the expressions for $\chi_{\nu\mu}$ and $\eta_{\nu\mu}$. In the radiation hydrodynamics problem we get the same expressions but with perturbations in all variables we solve for through the new expressions for $\delta\chi_{\nu\mu}$ and $\delta\eta_{\nu\mu}$. In the energy equation we treat the radiative transfer in the same way as in the rate equations.

In the diagnostic problem we could treat one atomic species at a time but here we need to include all the energy levels of all the elements that contribute to the energy balance or the electron density. With a six-level model for hydrogen and a six-level model for ionized calcium (taking into account the most important energy levels for the Solar chromosphere) we end up with 10 rate equations, 2 particle conservation equations and 4 hydrodynamic equations per depth-point. With 100 depth-points we total 1600 equations in 1600 unknowns. This set has to be solved for each time-step. To enable the treatment of shocks it is an advantage to formulate the equations on an adaptive depth-grid. This means that the grid is neither fixed relative to the solar center (Eulerian grid) nor relative to the moving fluid (Lagrangian grid) but something in between. The grid locations are solved for together with the other variables and we add one equation that describes where we want the grid-points to be. One may choose to set the grid density to be proportional to gradients in the variables. See Dorfi & Drury (1987) for details.

7.5 Dynamics

The classical assumption of hydrostatic equilibrium means there are no systematic velocity fields in the atmosphere. The balance between the pressure gradient and the gravity leads to an exponentially decreasing density with height. In the absence of damping, any acoustic waves excited are therefore bound to rapidly increase in amplitude and form shocks in the outer atmosphere; weak shock theory predicts this to happen at about 1Mm above the visible solar surface (Stein & Schwartz 1972). Any classical model atmosphere will therefore fail to describe the outer atmosphere — dynamic phenomena will quickly dominate and it will be *meaningless* to use mean quantities to describe the atmosphere.

There is a large literature on acoustic waves in stellar atmospheres, see reviews by Narain & Ulmschneider (1990, 1996) for references.

To properly describe the behaviour of waves in the chromosphere it is necessary with a fully consistent coupling between non-LTE radiation and the hydrodynamics. Such a consistent treatment was achieved only rather recently (Carlsson & Stein 1992, 1994, 1995, 1997a, 1997b) The equations were formulated along the lines given in the previous section. The starting atmosphere was taken from a static convective-radiative equilibrium model thus having no chromospheric temperature rise. As bottom boundary condition was taken a velocity field deduced from observations of the Doppler shift of an iron line formed around 280 km height (Lites et al. 1993). We will here only summarize some of the conclusions. For details, the reader is referred to the original papers.

It was found that statistical equilibrium at the instantaneous values of the hydrodynamic variables was a *bad* approximation. At times of compression in a wave the instantaneous values would give increased hydrogen ionization with the energy increase absorbed by an increase of the hydrogen ionization energy with only a small temperature increase as a consequence. Because of the long timescales for hydrogen ionization and recombination this is not a realistic picture. Hydrogen does not have time to ionize in the compression phase above about 500 km height and the energy therefore goes into increased temperature instead of into hydrogen ionization energy. This leads to a much sharper temperature increase over shock fronts than would be the case with infinitely fast ionization/recombination rates. See Carlsson & Stein (1992) for details.

Another conclusion from the simulation work is that our traditional picture of stellar chromospheres has to be radically rethought. Carlsson & Stein (1994, 1995) find that the temperature rise exhibited in semiempirical models of the non-magnetic solar chromosphere is mainly a result of non-linear averaging of a shock dominated atmosphere. It is here important to stress that *any* dynamic atmosphere will lead to an *overestimate* of the mean temperature when mean intensities in the blue are used as diagnostics. The simulations show a mean gas temperature that does *not* increase with height giving mean intensities that give a semi-empirical temperature rise (Fig. 5). The simulations do not claim to contain the full picture of the Solar chromosphere; *e.g.*, they lack the effect of an incident radiation field from the corona and high frequency waves from the photosphere. Both of these effects will favour a slow temperature increase. The essential conclusions are, however:

- a mean temperature structure can *not* be deduced from mean intensities in the blue part of the spectrum.
- in a dynamic chromosphere the mean temperature is not only difficult to deduce but is also a *meaningless* and even *misleading* quantity. The energy balance can only be deduced from a dynamical model and not from any such mean model.

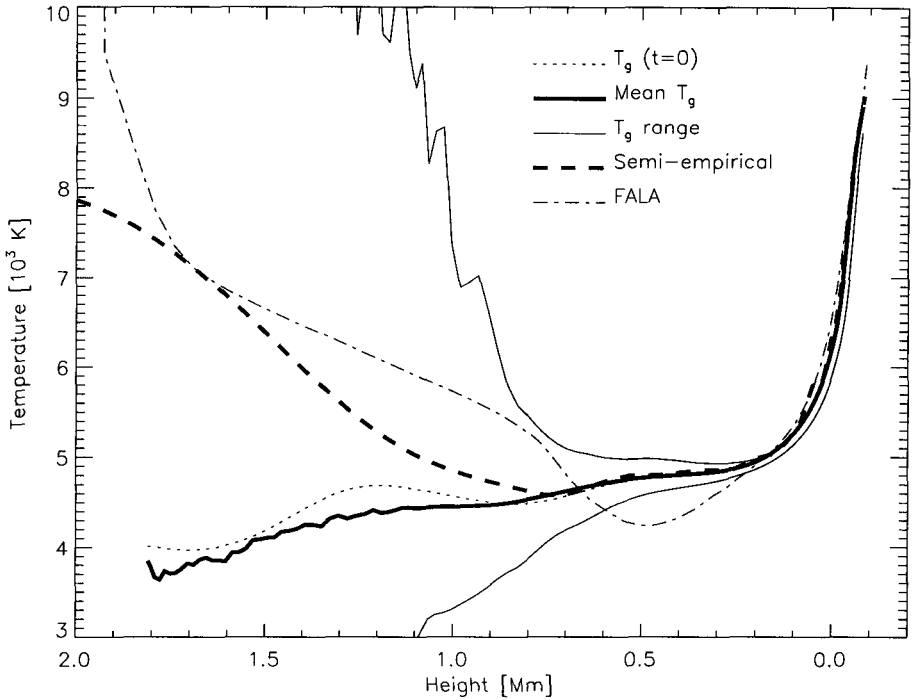


Fig. 5. Time average of the temperature in a dynamical simulation (Carlsson & Stein 1995), the range of temperatures in the simulation, the semiempirical model that gives the best fit to the time average of the intensity as a function of wavelength calculated from the dynamical simulation, the starting model for the dynamical simulation and the semiempirical model FALA (Fontenla et al. 1993). The maximum temperatures are only reached in narrow shock spikes of short duration. The semiempirical model giving the same intensities as the dynamical simulation shows a chromospheric temperature rise while the mean temperature in the simulation does not.

The diagnostics of the Solar chromosphere is made difficult by the lack of good observational diagnostics. In the optical region the continuum opacity is much too small and we thus need a large line opacity. This leaves lines originating from the ground state of the dominant ionization stage of abundant elements. Most of these resonance lines lie in the UV region of the spectrum with the notable exception of the two resonance lines from singly ionized calcium, Ca II. These Fraunhofer H and K lines show very strong absorption with a slight emission as a chromospheric signature near line center. The emission is in the form of two emission peaks with absorption in between. In non-magnetic regions outside the chromospheric network the emission is asymmetric with almost never a red emission peak and the violet peak (named H_{2V} for the H-line and K_{2V} for the K-line) only prominent at times with a

period of about three minutes. It has turned out that this asymmetry and the temporal behaviour is difficult to reproduce in theoretical models. Another way of seeing this difficulty is that the feature is a good diagnostic of the atmospheric conditions.

The simulations mentioned above do, however, reproduce the CaII H₂V behaviour; even in some detail, see Fig. 6.

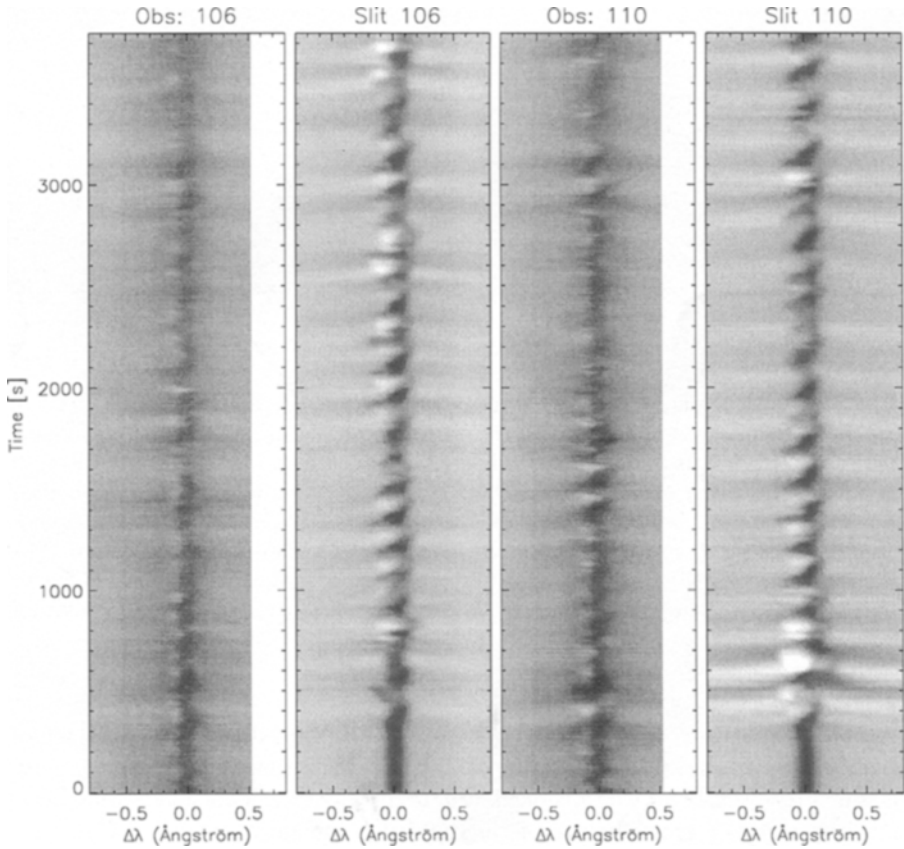


Fig. 6. The computed Ca II H line intensity as a function of wavelength and time compared with observations for slit positions 106 and 110. To mimic effects of scattered light and seeing the simulations have been convolved with a Gaussian with FWHM of 5 km/s, scattered light of 1% of the continuum intensity has been added and the sequence have been shifted in time with a random function sharply peaked at no shift. There is close correspondence between the simulated and observed grains after the initial startup period. From Carlsson & Stein (1997b).

The formation of the Ca II H_{2V} grains is explained in detail in Carlsson & Stein (1997a).

In the UV part of the spectrum we have access to more spectral features formed in the Solar chromosphere. All continua shortward of 152 nm are formed above 500 km height. With the SUMER spectrograph on-board the SOHO spacecraft this wavelength region is accessible. First results show that the non-magnetic chromosphere is indeed very dynamic. Continuum intensity variations are similar to the predictions from simulations (see Carlsson et al. 1997) while line emission variations are not. Emission lines from neutral elements show emission all the time while they sometimes disappear in the simulations. These lines are formed higher than the Ca II resonance lines and one explanation for the discrepancy may be that these layers are influenced by hotter material in magnetic flux-tubes that cross areas that lower down only have weak magnetic fields.

To further our understanding of the dynamics of the outer layers of the Solar atmosphere we need detailed observations coupled with detailed radiation-hydrodynamics simulations. We may then get enough constraints to construct a meaningful *dynamic* picture of the solar atmosphere. Anything short of a consistent dynamic picture has to be treated with utmost caution.

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