

Homework 1.

P. 34 2, 4

P. 51 1bc, 2cd, 3a,b, 5

P34

$$\begin{aligned}
 \#2 \quad a) \quad L(c_1 u_1 + c_2 u_2) &= \frac{\partial}{\partial x} \left[ k_o(x) \frac{\partial (c_1 u_1 + c_2 u_2)}{\partial x} \right] \\
 &= \frac{\partial}{\partial x} \left[ k_o(x) c_1 \frac{\partial u_1}{\partial x} + k_o(x) c_2 \frac{\partial u_2}{\partial x} \right] \\
 &= c_1 \frac{\partial}{\partial x} (k_o(x) \frac{\partial u_1}{\partial x}) + c_2 \frac{\partial}{\partial x} (k_o(x) \frac{\partial u_2}{\partial x}) \\
 &= c_1 L u_1 + c_2 L u_2 \quad \checkmark
 \end{aligned}$$

b). This is not a linear operator since  $k_o(x, c_1 u_1 + c_2 u_2)$  is not necessarily linear.

# 4. a)  $u_p$  satisfies  $L u_p = f$  $u_1, u_2$  satisfy  $L u_i = 0$ 

$$\begin{aligned}
 \Rightarrow L(u_p + c_1 u_1 + c_2 u_2) &= L u_p + c_1 L u_1 + c_2 L u_2 \\
 &= f
 \end{aligned}$$

$\Rightarrow u_p + c_1 u_1 + c_2 u_2$  is a particular soln.

b) If  $L u = f_1 + f_2$  s.t.  $L u_{p_1} = f_1$  and  $L u_{p_2} = f_2$ .

$$\begin{aligned}
 \text{Then } f_1 + f_2 &= L u_{p_1} + L u_{p_2} \\
 &= L(u_{p_1} + u_{p_2})
 \end{aligned}$$

$\Rightarrow u_{p_1} + u_{p_2}$  is a particular solution of  $L u = f_1 + f_2$ .

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(2)

#1 b.  $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - v_0 \frac{du}{dx}$

let  $u(x,t) = G(t)\phi(x)$ .

$$\Rightarrow G_t \phi = kG \phi_{xx} - v_0 G \phi_x$$

$$\Rightarrow \frac{G_t}{G} = \frac{k \phi_{xx} - v_0 \phi_x}{\phi} = -\lambda$$

$$\Rightarrow \frac{G_t}{G} = -\lambda \text{ and } k\phi_{xx} - v_0 \phi_x + \lambda \phi = 0$$

c)  $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

let  $u(x,y) = G(y)\phi(x)$ .

$$\Rightarrow G \phi_{xx} + G_y \phi = 0$$

$$\Rightarrow -\frac{G_{yy}}{G} = \frac{\phi_{xx}}{\phi} = -\lambda$$

$$\Rightarrow G_{yy} - \lambda G = 0 \text{ and } \phi_{xx} + \lambda \phi = 0$$

2 c.  $\frac{d^2 \phi}{dx^2} + \lambda \phi = 0$  s.t.  $\phi'(0) = 0$  and  $\phi'(L) = 0$

$$\lambda > 0 \quad \phi(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$\phi'(0) = 0 \Rightarrow c_2 = 0$$

$$\phi'(L) = 0 \Rightarrow \sqrt{\lambda} L = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2 \text{ and } \phi_n = \cos \frac{n\pi x}{L}.$$

$$x=0 \quad \phi(x) = c_1 x + c_2$$

$$\phi'(0) = 0 \Rightarrow c_2 = 0 \Rightarrow \phi(x) = c_1 x \text{ works}$$

$$\phi'(L) = 0 \text{ no info}$$

(3)

$$\lambda < 0 \quad \phi(x) = C_1 \cosh \sqrt{\lambda} x + C_2 \sinh \sqrt{\lambda} x$$

$$\phi'(0) = 0 \Rightarrow C_2 = 0$$

$$\phi'(L) = 0 \Rightarrow \sinh \sqrt{\lambda} L = 0 \Rightarrow \sqrt{\lambda} = 0 \rightarrow \leftarrow.$$

2d.  $\phi'' + \lambda \phi = 0$  with  $\phi(0) = 0, \phi'(0) = 0$

$$\lambda > 0 \quad \phi(x) = C_1 \cos \sqrt{\lambda} x + C_2 \sin \sqrt{\lambda} x$$

$$\phi(0) = 0 \Rightarrow C_1 = 0$$

$$\phi'(L) = 0 \Rightarrow \cos \sqrt{\lambda} L = 0$$

$$\Rightarrow \sqrt{\lambda} L = \frac{(2n+1)\pi}{2} \Rightarrow \lambda_n = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

$$\phi_n = \sin \frac{(2n+1)\pi}{2L} x$$

$$\lambda = 0 \quad \phi(x) = C_1 x + C_2$$

$$\phi(0) = 0 \Rightarrow C_2 = 0 \quad \text{trivial soln.}$$

$$\phi'(L) = 0 \Rightarrow C_1 = 0$$

$$\lambda < 0 \quad \phi(x) = C_1 \cosh \sqrt{\lambda} x + C_2 \sinh \sqrt{\lambda} x$$

$$\phi(0) = 0 \Rightarrow C_1 = 0$$

$$\phi'(L) = 0 \Rightarrow \cosh \sqrt{\lambda} L = 0 \quad \underline{\underline{\text{no soln.}}}$$

(4)

$$3 \text{ a } u(x, t) = 6 \sin \frac{9\pi x}{L} e^{-(\frac{9\pi}{L})^2 t}$$

$$\Rightarrow B_9 = 6 \Rightarrow u(x, t) = 6 \sin \frac{9\pi x}{L} e^{-\frac{(9\pi)^2}{L} t}$$

$$b. u(x, 0) = 3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L}$$

$$\Rightarrow B_1 = 3, B_3 = -1 e^{-(\frac{\pi}{L})^2 t}$$

$$\Rightarrow u(x, t) = 3 \sin \frac{\pi x}{L} e^{-\frac{(\pi)^2}{L} t} - \sin \frac{3\pi x}{L} e^{-\frac{(3\pi)^2}{L} t}$$

$$5. \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \quad n > 0, m > 0$$

$$\begin{aligned} &= \int_0^L \frac{1}{2} (\cos \frac{(n-m)\pi x}{L} - \cos \frac{(n+m)\pi x}{L}) dx \\ &= \frac{1}{2} \left[ \frac{L}{(n-m)\pi} \sin \frac{(n-m)\pi x}{L} - \frac{L}{(n+m)\pi} \sin \frac{(n+m)\pi x}{L} \right]_0^L = 0 \end{aligned}$$

if

$$n = m$$

$$\Rightarrow \int_0^L \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx = \frac{1}{2} \int_0^L (1 - \cos 2\frac{n\pi x}{L}) dx = 4_2$$