

HWK 6

(1)

P161 3, 5d, 8 P174 (e, c), 2, 9, 11.

P161

#3

$$\frac{d^2 \psi}{dx^2} + \alpha(x) \frac{d\psi}{dx} + [\lambda \beta(x) + \gamma(x)] \psi = 0$$

$$H(x) \frac{d^2 \psi}{dx^2} + \alpha(x) H(x) \frac{d\psi}{dx} + H(x) [\lambda \beta(x) + \gamma(x)] \psi = 0$$

To put this in standard S-L form we need to make it look like

$$P(x) \frac{d^2 \psi}{dx^2} + \frac{dP}{dx} \frac{d\psi}{dx} + [\lambda \sigma(x) + q(x)] \psi = 0$$

$$\Rightarrow \cancel{H(x) = \frac{dH}{dx}} \quad \boxed{H(x) = P(x)} \quad \int_0^x \alpha(x) dx$$

and $\frac{dH}{dx} = \alpha H(x) \Rightarrow H(x) = c e^{\int_0^x \alpha(x) dx}$

$$H(x) \beta(x) = \sigma(x) \Rightarrow \sigma(x) = c e^{\int_0^x \alpha(x) dx} \beta(x)$$

$$H(x) \gamma(x) = q(x) \Rightarrow q(x) = c e^{\int_0^x \alpha(x) dx} \gamma(x)$$

#5d.

$$\lambda = \frac{-p \phi \frac{d\phi}{dx} \Big|_0^L + \int_0^L P \left(\frac{d\phi}{dx} \right)^2 - q \phi^2}{\int_0^L \phi^2 dx}$$

where $\phi'(0) = \phi'(L) = 0$

$$\sigma(x) = 1 \Rightarrow \lambda = \frac{\int_0^L P \left(\frac{d\phi}{dx} \right)^2 dx}{\int_0^L \phi^2 dx} \geq 0$$

$p(x) = 1$
 $q(x) = 0$

Note at $\lambda = 0$ $\phi_0 = 1$ satisfies the B.C. and ODE

$$\Rightarrow \lambda_1 = 0 < \lambda_2 < \lambda_3 < \dots$$

8. $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \Rightarrow P(x) = 1 \quad q(x) = -x^2$
 $\sigma(x) = 1$

$\frac{d\phi}{dx}(0) = 0 \quad \frac{d\phi}{dx}(1) = 0$

using the R.P $\lambda = \frac{\int_0^1 \left(\frac{d\phi}{dx}\right)^2 - (-x^2)\phi^2 dx}{\int_0^1 \phi^2 dx} = \frac{\int_0^1 \left(\frac{d\phi}{dx}\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx} \geq 0$

Let's check $\lambda = 0$

$\Rightarrow \frac{d^2\phi}{dx^2} - x^2\phi = 0$

subject to $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(1) = 0$

In fact it would be hard to conclude anything from this

~~$\frac{d\phi}{dx} = 0$~~

But look at the R.P. For $\lambda = 0$ to be an eigenvalue, it would require $\frac{d\phi}{dx} = 0$ and $\phi = 0$ so that the denominator \Rightarrow But then ϕ cannot be an eigenfunction since it cannot be 0,

$\Rightarrow \lambda \neq 0$

P 194

(3)

1 a). $\phi(0) = \phi(L) = 0$ Dirichlet

$$\Rightarrow \int_0^L p \left(\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx} \right) dx = 0 \quad \text{by simple substitution}$$

b). $\phi'(0) = 0, \phi(L) = 0$

$$\begin{aligned} \Rightarrow \int_0^L p \left(\phi_1 \frac{d\phi_2}{dx} - \phi_2 \frac{d\phi_1}{dx} \right) dx &= p(L) \left[\cancel{\phi_1(L)} \frac{d\phi_2(L)}{dx} - \phi_2(L) \frac{d\phi_1(L)}{dx} \right] \\ &\quad - p(0) \left[\phi_1(0) \frac{d\phi_2(0)}{dx} - \cancel{\phi_2(0)} \frac{d\phi_1(0)}{dx} \right] \\ &= 0 \end{aligned}$$

2. $\frac{d}{dx} \left(p(x) \frac{d\phi}{dx} \right) + q(x)\phi + \lambda \sigma(x)\phi = 0$

where $\phi(1) = 0$

$$\phi(2) - 2 \frac{d\phi}{dx}(2) = 0$$

the equation is a regular S-L eq with B.C. where $\beta_1 = 1, \beta_2 = 0$

$\beta_3 = 1$ and $\beta_4 = -2$. $\Rightarrow \phi_n \perp \phi_m$ wrt. $\sigma(x)$

You can also directly show this by computation.

I don't see any way to explicitly calculate $\sigma(x)$.

9. $\frac{d^4 \phi}{dx^4} + \lambda e^x \phi = 0$

$\phi(0) = 0 \quad \phi(1) = 0$

$\frac{d\phi}{dx}(0) = 0 \quad \frac{d^2\phi}{dx^2}(1) = 0$

try multiplying by ϕ and integrating by parts.

$\int_0^1 \phi \frac{d^4 \phi}{dx^4} + \int_0^1 \lambda e^x \phi^2 = 0$

$\Rightarrow \lambda = - \frac{\int_0^1 \phi \frac{d^4 \phi}{dx^4} dx}{\int_0^1 e^x \phi^2 dx} = - \frac{\phi \phi'''' \Big|_0^1 - \int_0^1 \phi' \phi'''' dx}{\int_0^1 e^x \phi^2 dx}$

$= \frac{\phi' \phi^4 \Big|_0^1 - \int_0^1 (\phi''')^2 dx}{\int_0^1 e^x \phi^2 dx}$

Since the B.C. kill off the first term in numerator

$= - \frac{\int_0^1 (\phi''')^2 dx}{\int_0^1 e^x \phi^2 dx} \leq 0$

let's try $\lambda = 0$ case.

① $\phi'''' = 0 \Rightarrow \phi''' = c_1 \Rightarrow \phi'' = c_1 x + c_2 \Rightarrow \phi' = \frac{c_1}{2} x^2 + c_2 x + c_3$

$\Rightarrow \phi(x) = \frac{c_1}{6} x^3 + \frac{c_2}{2} x^2 + c_3 x + c_4$

$\phi(0) = 0 \Rightarrow c_4 = 0$

$\phi'(0) = 0 \Rightarrow c_3 = 0$

$\phi(1) = 0 \Rightarrow \frac{c_1}{6} + \frac{c_2}{2} + c_3 = 0$

$\phi''(1) = 0 \Rightarrow c_2 = 0$

\Downarrow
 $c_1 = 0 \Rightarrow c_2 = c_3 = 0$

So NO $\lambda \neq 0$.

$$4 \quad L = P(x) \frac{d^2}{dx^2} + r(x) \frac{d}{dx} + q(x)$$

5

$$\Rightarrow \int_a^b v L(u) dx = \int_a^b v [p u'' + r u' + q(x)u] dx$$

$$= \int_a^b p v u'' + r v u' + q v u dx$$

$$= p v u' + r v u \Big|_a^b - \int_a^b (p v)' u' + (r v)' u dx + \int_a^b q v u dx$$

the terms involve only u.

integrate by parts again

$$= p v u' + r v u \Big|_a^b - [(p v)' u] \Big|_a^b + \int_a^b (p v)'' u dx - \int_a^b ((r v)' + q v) u dx$$

$$= p v u' + r v u - (p v)' u \Big|_a^b + \int_a^b [p v'' + 2p' v' + p'' v - r' v - r v' + q v] u dx$$

$$= H(x) + \int_a^b [p v'' + (2p' - r') v' + (p'' - r' + q) v] u dx$$

$$= H(x) + \int_a^b u L^*(v) dx$$

where

$$L^*(v) = p \frac{d^2 v}{dx^2} + (2p' - r') \frac{dv}{dx} + (p'' - r' + q) v$$

Self adjoint if $H(x) = 0$

note that $H(x)$ is the same as back of the book except they differ by a negative sign because my eq is $\int v L(u) dx = H(x) - \int u L^*(v) dx$.