Math 481 - Hwk 1 Solutions

Section 5.1 # 11a, 12, 22, 23

11a. $S = \{(x_1, x_2, x_3, x_4) : |x_i| < 3 \quad i = 1,2,3\}$

Note x_4 is free in the above set

(i) $\partial S \neq \{(x_1, x_2, x_3, x_4) : |x_i| = 3 \quad i = 1,2,3\}$

These are all of the points on the vertices of the cube of side length 6 in $\mathbb{R}^3$.

Instead we need the boundary of 3 separate sets

$\partial S_1 = \{(x_1, x_2, x_3, x_4) : |x_1| = 3 \quad |x_2| \leq 3, |x_3| \leq 3\}$

$\partial S_2 = \{(x_1, x_2, x_3, x_4) : |x_1| \leq 3, |x_2| = 3, |x_3| \leq 3\}$

$\partial S_3 = \{(x_1, x_2, x_3, x_4) : |x_1| \leq 3 \quad |x_2| \leq 3 \quad |x_3| = 3$

$\Rightarrow \partial S = \partial S_1 \cup \partial S_2 \cup \partial S_3$

Note

$\neq \{(x_1, x_2, x_3, x_4) : |x_i| \leq 3 \quad i = 1,2,3\}$ since this contains interior to the cube

(ii) $\bar{S} = S \cup \partial S$

$= \{(x_1, x_2, x_3, x_4) : |x_i| \leq 3, \quad i = 1,2,3\}$

(iii) $S^o = S$ since S is open

(iv) exterior of S = interior of S^c

$= \{(x_1, x_2, x_3, x_4) : |x_i| > 3 \quad i = 1,2,3\}$, for at least one $i\}$

note x_4 is still free.

12. a) $S = \{(x_1, x_2, x_3, x_4) : |x_1| > 0 \quad x_2 < 1, x_3 \neq -2\}$

This is an open set since every one of its points is an interior point.

b) $S = \{(x_1, x_2, x_3, x_4) : x_1 = 1 \quad x_3 \neq -4\}$

This is neither open nor closed.

Not open since ~~xpa~~ $x = (1, 0, 0, 0) \in S$ but $x \notin S^o$.

Not closed since $x = (1, 0, -4, 0)$ is a limit point of S but is not contained in S

c. $S = \{(x_1, x_2, x_3, x_4) : x_1 = 1 \quad -3 \leq x_2 \leq 1 \quad x_4 = -5\}$

S is a closed set since S^c is open or by seeing that $S = \bar{S}$

22. Let $S_1 \supset S_2 \supset \dots S_m \supset \dots$ s.t. $S_i \neq \phi$, each S_i is compact. We need to show that $\bigcap_{m=1}^{\infty} S_m$ is non-empty.

Also we will show that the conclusion does not follow if we assume each S_i to be closed. Let's do that first.

observe $S_n = [n, \infty)$ is closed and $S_n \supset S_{n+1}$ $\forall n$

but $\bigcap_{n=1}^{\infty} S_n = \phi$.

3 To show why compactness allows $\bigcap_{n=1}^{\infty} S_n \neq \phi$

Let's first use B-W Thm. which says that every bounded infinite set of points in $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Let $x_i \in S_i$ and $x_i \neq x_j$ if $i \neq j$

Then $\{x_i\}$ is a bounded sequence of pts since each S_i is bounded (since each is compact).

Thus by B-W $\exists$ $\bar{x}$ which is a limit point of $\{x_i\}$

Next we can assume each S_i has infinitely many points, because if not, then $\exists$ k s.t. S_k has finitely many points and so would S_{k+1}, S_{k+2} etc. Thus some point from S_k must lie in all subsequent S_{k+1} $\Rightarrow \bigcap S_k \neq \phi$.

So if each S_i is also infinite, we claim that $\bar{x}$ is a limit point of each S_i

Why? Since $S_n \supset S_{n+1}$ then an infinite subsequence of $\{x_n\}$ lies in each S_i. Thus $\bar{x}$ is a limit point of each of these subsequences

Since each S_i is closed $\bar{x} \in S_i$ $\forall i$

$$\Rightarrow \bar{x} \in \bigcap_{i=1}^{\infty} S_i$$

which is therefore non-empty

23. Suppose $U_1, U_2 \ldots$ open sets covers compact set S

show $S \subset \bigcup_{m=1}^{N} U_m$ for some N. That is show that

a finite subcover of S exists.

Consider $S_n = \cancel{A}\, S \cap \left(\bigcup_{m=1}^{n} U_m \right)^c$

Then $S_1 = S \cap U_1^c$

$S_2 = S \cap (U_1 \cup U_2)^c$ $\Rightarrow S_1 \supset S_2$

It's straightforward to see that in general

$S_n \supset S_{n+1}$. If each S_n is nonempty then

by #22 $\bigcap_{n=1}^{\infty} S_n$ is non-empty (*)

But since the U_i cover S, i.e. $S \subset \bigcup_{i=1}^{\infty} U_i$

$\Rightarrow S \cap \left(\bigcup_{i=1}^{\infty} U_i \right)^c = \phi$

but this contradicts (*)

$\Rightarrow$ At least one S_M is empty for finite M.

$\Rightarrow S \subset \bigcup_{m=1}^{M} U_m$.