Midterm I Solutions

1. a) $S = \{(x,y,z) : x^2+y^2 < z \,;\, 0 < z \le 2\}$

$\partial S = \{(x,y,z) : x^2+y^2 = z \quad 0 \le z \le 2\} \cup \{(x,y,z) : x^2+y^2 \le z, z=2\}$

$\bar{S} = S \cup \partial S = \{(x,y,z) : x^2+y^2 \le z \quad 0 \le z \le 2\}$

$S^\circ = \{(x,y,z) : x^2+y^2 < z : 0 < z < 2\}$

exterior of S = interior of S^c

$= \{(x,y,z) : \underset{0<z<2}{x^2+y^2 > z}\} \cup \{(x,y,z) : z > 2 \text{ or } z < 0\}$

b) $S = \{(x,y) : x \in \mathbb{Q} \cap [0,1], y \in \mathbb{Q}^c \cap [0,1]\}$

$\partial S = \{(x,y) : x \in [0,1], y \in [0,1]$

$\bar{S} = \partial S$

$S^\circ = \emptyset$

$(S^c)^\circ = \{(x,y) : -\infty < x < 0 \text{ or } 1 < x < \infty\} \cup \{(x,y) : \begin{matrix} -\infty < y < 0 \\ 1 < y < \infty \end{matrix}\}$

2. Fix $\varepsilon > 0$, and consider $f(x,y) = 1 + 2x^2 - y^2$

$|1 + 2x^2 - y^2 - 2| = |2x^2 - y^2 - 1|$

$= |2(x^2-1) - (y^2-1)|$

$= |2(x-1)(x+1) - (y-1)(y+1)|$

Now suppose that $(x-1)^2 + (y-1)^2 \le 1$

$\Rightarrow \quad x \le 2$

$\quad\quad y \le 2$

$\Rightarrow$

$$|1+2x^2-y^2-2| = |2(x-1)(x+1) - (y-1)(y+1)|$$

$$\leq |2\cdot 3(x-1) + 3(y-1)|$$

$$\leq |(6,3)||(x-1, y-1)|$$

$$= \sqrt{45}\,|\bar{x}-\bar{x}_0| \qquad \bar{x} = (x,y),\ \bar{x}_0 = (1,1)$$

$\Rightarrow$ if $|\bar{x}-\bar{x}_0| < \frac{\varepsilon}{\sqrt{45}} = \delta$ the result we want holds

Thus we choose $\delta = \min(1, \varepsilon/\sqrt{45})$ in which case

$|\bar{x}-\bar{x}_0| < \delta \Rightarrow |f(\bar{x}) - L| < \varepsilon$.

3. $f(x,y) = \begin{cases} (x-y)^2 \sin \frac{1}{x-y} & x \neq y \\ 0 & x = y \end{cases}$

If $x \neq y$ then $f_x = 2(x-y)\sin\frac{1}{x-y} - \cancel{(x-y)^2}\cos\frac{1}{(x-y)}\cdot\frac{1}{\cancel{(x-y)^2}}$

Note $\lim_{y\to x} f_x$ DNE for this case

But if $x=y$

then $f_x(x,x) = \lim_{h\to 0}\frac{f(x+h,x) - f(x,x)}{h} = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin{1/h} = 0$

So f_x exists $\forall (x,y)$ but is not continuous on $x=y$.

4. $S_n = \{(x,y): x^2+y^2 \leq \frac{1}{n}, n \geq 1\}$

Note $S_1 = \{(x,y): x^2+y^2 \leq 1\}$ which are concentric circles

$S_2 = \{(x,y): x^2+y^2 \leq 1/2\}$

Thus it is easy to see that $S_n \supset S_{n+1}$

Each S_n is in fact compact, i.e. closed and bounded

and $(0,0) \in S_n \ \forall n \Rightarrow S_n \neq \phi \ \forall n$

Finally $\operatorname{diam} S_n = \frac{2}{\sqrt{n}} \to 0$

Thus all the conditions of the nested set theorem are satisfied.

and so $\bigcap_{n=1}^{\infty} S_n$ is a single point $= \{(0,0)\}$.

5. f is 2-valued on S if $f(x) \in T = \{0,1\}$ $\forall x \in S$ and f must be continuous on S.

We want to prove that if S is a connected subset of $\mathbb{R}^n$, then every 2-valued function on S is constant.

Let us assume that S is connected and $S = A \cup B$ where

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases}$$

We claim that both A and B are open sets.

If $x_0 \in A$ then by continuity $\exists \delta_A > 0$ s.t. $|x - x_0| < \delta_A \Rightarrow f(x) = 0$

Similarly if $x_0 \in B$ then by continuity $\exists \delta_B > 0$ s.t. $|x - x_0| < \delta_B \Rightarrow f(x) = 1$

In other words a nbhd of a point in A is contained in A and similarly for B.

But S is connected thus for $S = A \cup B$ where A and B open, disjoint sets

can only occur if either A or B is empty.

In either case f would be constant on S.