Math 480 - Midterm 1 Solutions

1. Let $S = [0,1) \cap \mathbb{Q}$ where $\mathbb{Q}$ is the set of rationals.

   a) Interior points: ϕ, since if $x_0 \in S$ then $B_\varepsilon(x_0) \not\subset S$ for any $\varepsilon > 0$.

   b) Limit points: $T = [0,1]$ since if $x_0 \in T$ then $(B_\varepsilon(x_0) - x_0) \cap S \neq \phi$.

   c) Exterior points: Interior points of S^c. $S^c = (-\infty, 0) \cup \{[0,1] \cap \mathbb{Q}^c\} \cup [1, \infty)$

   Thus the interior points are $T = (-\infty, 0) \cup (1, \infty)$

   d) Boundary points: $T = [0,1]$ since ~~for~~ if $x_0 \in T$, then $B_\varepsilon(x_0)$ contains points within both S and S^c

2. Prove by induction that $\forall n \geq 1$ $\quad 1+4+7+\dots+(3n-2) = \dfrac{n(3n-1)}{2}$

   Let P_n be the proposition

   Base Case: $n = 1$ $\quad LHS = 1$

   $$RHS = \frac{1(3-1)}{2} = 1 \quad \Rightarrow P_1 \text{ holds}$$

   Suppose P_k holds, that is $1+4+7+\dots+(3k-2) = \dfrac{k(3k-1)}{2}$

   We will show that P_{k+1} holds:

   $$\begin{aligned}
   1+4+7+\dots+(3k-2)+(3(k+1)-2) &= [1+4+\dots+(3k-2)] + 3(k+1)-2 \\
   &= \frac{k(3k-1)}{2} + 3(k+1)-2 \\
   &= \frac{3k^2-k}{2} + 3k+1 \\
   &= \frac{3k^2-k+6k+2}{2} \\
   &= \frac{(k+1)(3k+2)}{2} \\
   &= \frac{(k+1)(3(k+1)-1)}{2}
   \end{aligned}$$

   $\Rightarrow P_{k+1}$ holds

   $\Rightarrow P_n$ holds $\forall n \geq 1$.

3. Prove: $\lim_{x\to 1} x^4+1=2$

Fix $\varepsilon>0$, then $|x^4+1-2| = |x^4-1|$

$= |x^2-1||x^2+1|$

$= |x-1||x+1||x^2+1|$

$< |x-1|\cdot 3\cdot 5$

$< 15\delta_1$

$< \varepsilon$

Let $|x-1|<1$

$\Rightarrow$ $0<x<2$

$1<x+1<3$

$1<x^2+1<5$

if $|x-1| = \delta_1 < \varepsilon/15$

Thus if we choose $\delta=\min(\delta_1, 1)$ then the above result holds

Thus $\lim_{x\to 1} x^4+1=2$

4. See Book - Theorem 2.2.8 on Page 62.

5. $S = \{\text{negative integers}\} \cup \{2,4,6,8,\ldots\}$

S is countable since it is the countable union of countable sets

To align its elements with $\mathbb{N}$, we reorder

$$S = \{-1, 2, -2, 4, -3, 6, -4, 8, \ldots\}$$
$$\Updownarrow \quad \Updownarrow$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots\}$$

Thus let $f(x) = \begin{cases} x & \text{if } x \text{ is positive} \\ 2|x|-1 & \text{if } x \text{ is negative} \end{cases}$

There are other ways to align the set, so the above answer is not unique.

6. If $T(\theta)$ denotes the temperature along the circle such that $T(0) > T(\pi)$. Note $T(\theta)$ is a continuous function of θ. Further $T(0) = T(2\pi)$

Define $f(\theta) = T(\theta) - T(\theta+\pi)$

Note f is continuous since T is.

$$f(0) = T(0) - T(\pi) > 0$$

$$f(\pi) = T(\pi) - T(2\pi)$$
$$= T(\pi) - T(0) < 0$$

$\Rightarrow$ IVT implies $\exists\ \theta^*$ s.t. $f(\theta^*) = 0$

$\Rightarrow T(\theta^*) = T(\theta^* + \pi)$ for at least one value θ^*.