Math 480 midterm 2 solutions

1. If f and g are integrable on $[a,b]$, show that $3f-5g$ is as well and that $\int_a^b (3f-5g)(x)\,dx = 3\int_a^b f(x)\,dx - 5\int_a^b g(x)\,dx$

Any Rieman sum of $3f-5g$ over a partition P of $[a,b]$ can be written as

$$\sigma_{3f-5g} = \sum_{j=1}^{n} \left(3f(c_j) - 5g(c_j)\right)(x_j - x_{j-1})$$
$$= 3\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}) - 5\sum_{j=1}^{n} g(c_j)(x_j - x_{j-1})$$
$$= 3\sigma_f - 5\sigma_g.$$

Since f and g are integrable $\forall \varepsilon>0$, $\exists\, \delta_1, \delta_2 > 0$ s.t.

$$\left|\sigma_f - \int_a^b f(x)\,dx\right| < \varepsilon/6 \quad \text{if } \|P\| < \delta_1$$

$$\left|\sigma_g - \int_a^b g(x)\,dx\right| < \frac{\varepsilon}{10} \quad \text{if } \|P\| < \delta_2$$

Thus if $\|P\| < \delta = \min(\delta_1, \delta_2)$ then

$$\left|\sigma_{3f-5g} - 3\int_a^b f(x)\,dx + 5\int_a^b g(x)\,dx\right| = \left|3\sigma_f - 3\int_a^b f(x)\,dx - \left(5\sigma_g - 5\int_a^b g(x)\,dx\right)\right|$$
$$\le 3\left|\sigma_f - \int_a^b f(x)\,dx\right| + 5\left|\sigma_g - \int_a^b g(x)\,dx\right|$$
$$< 3\cdot \varepsilon/6 + 5\cdot \varepsilon/10$$
$$= \varepsilon.$$

2. Let $F(x) = \int_0^x e^{-t^2}\,dt$ on $[0,1]$.

a) $f(x) = e^{-x^2}$ is integrable on $[0,1]$ since it is continuous
see Theorem 3.2.8 on page 134.

b). Fix $\varepsilon > 0$ and note that $|e^{-x^2}| \le M$ on $[0,1]$

Then $|F(x) - F(y)| = \left|\int_y^x e^{-t^2}\,dt\right| \le M|x-y|$

$\Rightarrow$ $F(x)$ is continuous $\qquad < \varepsilon$ if $|x-y| < \delta = \frac{\varepsilon}{M}$.

c) Observe that $\frac{1}{x-x_0}\int_{x_0}^x e^{-x_0^2}\,dt = e^{-x_0^2} = f(x_0)$

fix $\varepsilon > 0$

$$\Rightarrow \left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \frac{1}{|x-x_0|}\left|\int_{x_0}^x f(t) - f(x_0)\,dt\right|$$

$$< \frac{\varepsilon \int_{x_0}^x dt}{|x-x_0|} \quad \text{by continuity of } f(x)$$

$$= \frac{\varepsilon|x-x_0|}{|x-x_0|}$$

$$= \varepsilon.$$

$\Rightarrow$ $F(x)$ is differentiable and $F'(x_0) = f(x_0)$.

(3) Use Riemann sums to show that $\int_0^1 x\,dx = 1/2$.

See book page 115, Example 3.1.2

(4) $f(x) = \begin{cases} 1 & x \in \mathbb{Q}^c \cap [0,1] \\ 0 & x \in \mathbb{Q} \cap [0,1] \end{cases} \Rightarrow \overline{\int_0^1} f(x)\,dx = 1$

$$\underline{\int_0^1} f(x)\,dx = 0$$

5. $\Delta f(x) = f(x+h) - f(x)$

We use Taylor's Theorem to estimate the error in the approximation $f'(x_0) \simeq \frac{\Delta f(x_0)}{h}$ on (x_0, x_0+h)

Since $f''(x)$ exists on (x_0, x_0+h) we can write

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(c)}{2}h^2 \qquad c \in (x_0, x_0+h).$$

$$\Rightarrow \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{f''(c)}{2}h$$

$$\Rightarrow \left| \frac{\Delta f(x_0)}{h} - f'(x_0) \right| = \left| \frac{f''(c)}{2}h \right|$$

$$\leq \frac{Mh}{2} \qquad \text{for } |f''| \leq M.$$