Hwk 1 Solutions

Section 1.1. 3, 8, 11

1.2, 2, 12

3. Show that $\sqrt{2}$ is irrational

Pf, Assume not, then $\exists\, p, q \in \mathbb{Z}$ s.t $\sqrt{2} = p/q$

where p and q have no common divisors.

$$\Rightarrow \quad 2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow p^2 \text{ is even}$$

But p^2 even $\Rightarrow p$ is even because if not $p = 2k+1 \Rightarrow p^2 = 4k^2 + 4k + 1$ is odd $\rightarrow \leftarrow$.

Since p is even, we can express it as $p = 2m$

$$\Rightarrow \quad (2m)^2 = 2q^2$$

$$\Rightarrow \quad q^2 = 2m^2$$

$$\Rightarrow \quad q^2 \text{ even and as above so is } q.$$

Thus both p and q have a common divisor of 2

which contradicts are assumption of no common divisor

$\Rightarrow \sqrt{2}$ is irrational

8. Let S and T be non-empty sets of real numbers s.t. every number is in S or T. If $s \in S$ and $t \in T$ then $s < t$. Prove there exists a unique $\beta \in \mathbb{R}$ s.t. every real number less than β is in S and every real number β is in T. (β is called a Dedekind cut).

Pf/ Note that any $t \in T$ is an upper bound for S, which is non-empty $\Rightarrow$ $\beta = \sup S$ exists and is unique

Similarly any $s \in S$ is a lower bound for T, which is also non-empty $\Rightarrow$ $\alpha = \inf T$ exists and is unique

Let $x \in \mathbb{R}$. Then by the order relationship $x < \beta$, $x = \beta$ or $x > \beta$.

Suppose $x < \beta$ and that $x \in T$. This would imply that x is an upper bound of S, which contradicts $\beta = \sup S$.

$\Rightarrow x \notin T$

$\Rightarrow x \in S$.

Similarly if $x > \beta$ and $x \in S$ we can show that this contradicts the definition of $\alpha = \inf T$

~~$\alpha = \inf T$~~

$\Rightarrow x \notin S$ and $x \in T$

Thus every real number less than β is in S and every real number greater than β is in T.

The statement also implies that $\exists x = \beta$ which is the value of the Dedekind Cut. The statement leaves ambiguous whether $\beta \in S$, T or neither.

11. Let S and T be non-empty sets of reals and let (3)

$$S-T = \{s-t : s\in S \text{ and } t\in T\}$$

a) Show that if S and T are bounded then

$$\sup(S-T) = \sup S - \inf T \quad (A)$$

and $\inf(S-T) = \inf S - \sup T \quad (B)$.

b) (A) and (B) hold for the extended reals.

Pf/ a) let's show (A)

Since S and T are bounded non-empty set $\beta_S = \sup S$, $\alpha_T = \inf T$ ~~exist~~ exist.

further if $x \in S-T$

then $x \le \sup S - \inf T = \beta_S - \alpha_T$

$\Rightarrow$ $S-T$ is bdd above

Further $S-T$ is nonempty since if $S=T \Rightarrow 0 \in S-T$

and if $S \neq T \Rightarrow$ at least one non-zero element exists in $S-T$.

$\Rightarrow$ β_{S-T} exists.

Clearly $\beta_{S-T} \le \beta_S - \alpha_T$ by above.

Now to show $\beta_{S-T} \ge \beta_S - \alpha_T$ consider $\varepsilon > 0$. By the definitions of β_S and α_T, $\exists$ $s_0 \in S$ and $t_0 \in T$ s.t.

$$\begin{matrix} \beta_S - \varepsilon/2 \le s_0 \\ t_0 \le \alpha_T + \varepsilon/2 \end{matrix} \Rightarrow \beta_S - \alpha_T - (s_0 - t_0) \le \varepsilon \text{ for arbitrary } \varepsilon.$$

$\Rightarrow \beta_S - \alpha_T \le s_0 - t_0 \le \beta_{S-T}$.

$\Rightarrow$ (A) holds.

I'll leave the proofs for (B) and (b) up to you.

Section 1.2

2. Prove $1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$ (P_n).

Base case P_1: $LHS = 1^2 = 1$

$RHS = \frac{1(1+1)(2+1)}{6} = 1 \implies P_1$ is true

Assume P_k holds for some k, i.e. $1^2+2^2+\cdots+k^2=\frac{k(k+1)(2k+1)}{6}$

then

$$1^2+2^2+\cdots+k^2+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$=(k+1)\left[\frac{k(2k+1)}{6}+k+1\right]$$

$$=\frac{k+1}{6}\left[2k^2+k+6k+6\right]$$

$$=\frac{k+1}{6}\left[2k^2+7k+6\right]$$

$$=\frac{(k+1)(k+2)(2k+3)}{6}$$

$$=\frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

$\implies P_{k+1}$ holds

Thus P_n hold $\forall n$ by Principle of Induction

12 For what integers is $\frac{1}{n!} > \frac{8^n}{(2n)!}$. Prove by induction. 5

By trial and error $n=7$ is a base case

assume P_k for $k>7$, i.e. $\frac{1}{k!} > \frac{8^k}{(2k)!}$

then

P_{k+1} is $\frac{1}{(k+1)!} = \frac{1}{(k+1)\,k!}$

$$> \frac{1}{k+1} \cdot \frac{8^k}{(2k)!}$$

$$= \frac{1}{k+1} \cdot \frac{8^{k+1}}{8} \cdot \frac{1}{(2k)!}$$

$$= \frac{8^{k+1}}{[2(k+1)]!} \cdot \frac{1}{8} \cdot \frac{1}{k+1} \cdot (2k+2) \cdot (2k+1)$$

$$> \frac{8^{k+1}}{[2(k+1)]!} \quad \text{since } \frac{(2k+2)(2k+1)}{8(k+1)} > 1$$

for $k>7$

$\Rightarrow P_{k+1}$ holds

Thus P_n is true for all $n \geq 7$.