# Hwk 3 Solutions



2.1 # 4c, 5, 11

2.2 # 7, 9, 10, 15a, 22a

## Section 2.1

4.c. Find $\lim_{x \to 0} \frac{1}{x^2-1}$ using an $\varepsilon$-$\delta$ proof

Here the limit $L = -1$.

Fix $\varepsilon > 0$. We want to show that $\left| \frac{1}{x^2-1} - (-1) \right| < \varepsilon$ if $|x-0| < \delta$.

Consider

$$\left| \frac{1}{x^2-1} - (-1) \right| = \left| \frac{x^2}{x^2-1} \right|$$

assume initially that $|x| < 1/2$

$\Rightarrow 3/4 \le |x^2-1| \le 1$

$$\le \frac{4}{3}|x^2|$$

$\Rightarrow 1 \le \frac{1}{|x^2-1|} \le \frac{4}{3}$

$$< \frac{4}{3}\delta^2 \quad \text{if } |x| < \delta$$

Thus we choose $\frac{4}{3}\delta^2 = \varepsilon$ or $\delta = \left(\frac{3}{4}\varepsilon\right)^{1/2}$ or $1/2$, whichever is smaller

5. Prove that $\lim_{x \to x_0} f(x) = L$ according to Def 2.1.2 with $|f(x) - L| < \varepsilon$

if and only if $\lim_{x \to x_0} f(x) = L$ with the new definition of $|f(x) - L| < k\varepsilon$

for any $k > 0$

Pf/

$\Rightarrow$ Fix $\varepsilon > 0$ and assume $\exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $|x - x_0| < \delta$.

Then note $|f(x) - L| < \varepsilon$

$< k\varepsilon$, if $k > 1$

if $k < 1$, let $\varepsilon_1 = k\varepsilon$.

Then $\exists\, \delta_1 > 0$ s.t. $|f(x) - L| < \varepsilon_1$

$= k\varepsilon$.

$\Leftarrow$ fix $\varepsilon > 0$

If $|f(x) - L| < K\varepsilon$ whenever $|x - x_0| < \delta$

then $|f(x) - L| < \varepsilon$ if $K > 1$.

If $K < 1$ let $\varepsilon_1 = \varepsilon/K$ and then choose $\delta_1$ s.t. $|x - x_0| < \delta_1$.

$$\Rightarrow |f(x) - L| < K\varepsilon_1 = K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

11. Prove: If $\lim\limits_{x \to x_0} f(x) = L > 0$, then $\lim\limits_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$

Fix $\varepsilon > 0$. Since $\lim\limits_{x \to x_0} f(x) = L \Rightarrow \exists\, \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Now consider

$$\left|\sqrt{f(x)} - \sqrt{L}\right| = \frac{\left|\sqrt{f(x)} - \sqrt{L}\right|\left|\sqrt{f(x)} + \sqrt{L}\right|}{\left|\sqrt{f(x)} + \sqrt{L}\right|}$$

$$= \frac{|f(x) - L|}{\left|\sqrt{f(x)} + \sqrt{L}\right|}$$

$\sqrt{L} < \sqrt{L} + \sqrt{f(x)}$

$\Rightarrow \frac{1}{\sqrt{L} + \sqrt{f(x)}} < \frac{1}{\sqrt{L}}$

$$< \frac{|f(x) - L|}{\sqrt{L}}$$

$$< \frac{\varepsilon}{\sqrt{L}}$$ since ~~$\sqrt{f(x)}$~~ $|f(x) - L| < \varepsilon$.

Now use the result of ~~#~~ Problem 5

to conclude that $\lim\limits_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$.

7. Let $f(x)=0$ if $x \in \mathbb{Q}^c$ and $f(p/q)=1/q$ if $p,q \in \mathbb{Z}^+$ with no common factors. Show that $f$ is discontinuous at every rational but continuous at every irrationals.

Pf/. The easier part is to show that $f$ is discontinuous at the rationals. Let $x_0 \in \mathbb{Q}$ where $x_0 = p/q$. Choose $\varepsilon < 1/q$. We know we can find an irrational, $x_{irr}$, arbitrarily close to $x_0$. Thus $|f(x_0) - f(x_{irr})| = |1/q - 0| > \varepsilon$
$\Rightarrow$ $f$ is discontinuous at rationals.

To show $f$ is continuous at irrationals, Fix $\varepsilon > 0$ and consider
$S = \{q : q \in \mathbb{Z}^+ \text{ and } \frac{1}{q} > \varepsilon\}$. Note that $\frac{1}{q} > \varepsilon \Rightarrow q < 1/\varepsilon$.
So for fixed $\varepsilon$, there are a finite number of integers $q$ less than $1/\varepsilon$.
Thus $S$ is a finite set.

Now let $y \in \mathbb{Q}^c$ and consider an interval that contains $y$, say $[y-1, y+1]$.
Choose $q \in S$ and consider $T_q = \{p : \frac{p}{q} \in [y-1, y+1], p \in \mathbb{Z}^+\}$.
By the archimedean property $T_q$ is also a finite set for each $q$.
Thus the set of rationals that lie in $[y-1, y+1]$ is finite.
To see this let $V = \{\frac{p}{q} : q \in S, p \in T_q\}$ which represents all such rationals and is clearly finite since $S$ and $T_q$ are.

Let $d$ = minimum value from $v \in V$ to $y$. Note $d > 0$ by finiteness.

Now let $x \in \mathbb{Q}$ s.t. $|y - x| < d \Rightarrow x \notin V$ ~~where~~ if $x = p/q$, $1/q < \varepsilon$

$$\Rightarrow |f(y) - f(x)| = |f(x)| = 1/q < \varepsilon.$$

~~Both continuous~~ if $x \in \mathbb{Q}^c \Rightarrow |f(y) - f(x)| = 0 < \varepsilon$

$\Rightarrow$ $f$ is continuous at all irrationals.

9. The characteristic function $\Psi_T$ of a set $T$ is

$$\Psi_T(x) = \begin{cases} 1 & x \in T \\ 0 & x \notin T. \end{cases}$$

Show that $\Psi_T$ is continuous at a point $x_0$ iff $x_0 \in T^0 \cup (T^c)^0$

$\Rightarrow$ assume $\Psi_T$ is continuous at a point $x_0$.

~~suppose~~, Then $\exists\ \forall \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. If we choose $\varepsilon = 1/2$ then the only way for $|f(x) - f(x_0)| < 1/2$ is if $f(x) = f(x_0)$ $\therefore$ Thus $\forall$ ~~all~~ $x \in B_\delta(x_0)$ $f(x) = f(x_0)$. This implies that $B_\delta(x_0)$ lies entirely within $T$ or $T^c$

Thus $x_0 \in T^0$ or $(T^c)^0 \Rightarrow x_0 \in T^0 \cup (T^c)^0$

$\Leftarrow$ Assume $x_0 \in T^0 \cup (T^c)^0$, Let's assume $x_0 \in T^0$, then $x_0$ is an interior point of $T$. Thus $\exists\ \delta > 0$ s.t. $B_\delta(x_0) \subset T$, Thus if $x \in B_\delta(x_0)$, $f(x) = 1$ and since $f(x_0) = 1 \Rightarrow |f(x) - f(x_0)| = 0 < \varepsilon$ for any $\varepsilon > 0$

A similar proof shows that if $x_0 \in (T^c)^0$, then $\exists\ \delta_1 > 0$ st $B_{\delta_1}(x_0) \subset T^c$ and $x \in B_{\delta_1}(x_0) \Rightarrow |f(x) - f(x_0)| = |0 - 0| = 0 < \varepsilon$.

10. Prove: If $f$ and $g$ are cont on $(a,b)$ and $f(x) = g(x)$ for every $x$ in a dense subset of $(a,b)$, then $f(x) = g(x)\ \forall x \in (a,b)$

Pf/ Let $S$ be a dense subset of $(a,b)$ and $T = (a,b) - S$ the set of points in $(a,b)$ not in $S$.

If $x_0 \in (a,b)$ then $x_0 \in S$ or $x_0 \in T$ since $S \cup T = (a,b)$ and $S \cap T = \emptyset$

If $x_0 \in S$, then by ~~defn~~ assumption $f(x_0) = g(x_0)$

So let's assume $x_0 \in T$ and show that $f(x_0) = g(x_0)$.

Let $h(x) = f(x) - g(x)$.

$h$ is continuous since $f$ and $g$ are. In particular at $x_0$

$\Rightarrow \forall \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |h(x) - h(x_0)| < \varepsilon$.

Since $S$ is dense in $(a,b)$ every nbhd of $x_0 \in T$ contains points in $S$

Thus for any $\delta > 0$ $B_\delta(x_0)$ contains points in $S$

let $x \in S \cap B_\delta(x_0)$.

Then $|h(x) - h(x_0)| < \varepsilon \Rightarrow |0 - h(x_0)| < \varepsilon$ (note $h(x) = 0$ since $f(x) = g(x)$ for $x \in S$.)

$\Rightarrow |h(x_0)| < \varepsilon$

Thus if $x_0 \in T$, $h(x_0)$ is arbitrarily small

$\Rightarrow h(x_0) = 0$

$\Rightarrow f(x_0) = g(x_0)$ if $x_0 \in T$

$\Rightarrow f(x) = g(x)$ $\forall x \in (a,b)$.

15 a) Prove: If $f$ is continuous at $x_0$ and $f(x_0) > \mu$, then $f(x) > \mu$ for all $x$ in some nbhd of $x_0$.

Since $f$ is continuous at $x_0$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Now choose $\varepsilon > 0$ s.t. $\mu + \varepsilon < f(x_0)$. Thus $\exists \delta > 0$ s.t.

$|f(x) - f(x_0)| < \varepsilon$ if $x \in B_\delta(x_0)$

$\Rightarrow f(x_0) - \varepsilon < f(x)$ ↓ since $\mu < f(x_0) - \varepsilon$

$\Rightarrow \mu < f(x)$ if $x \in B_\delta(x_0)$

Thus $\exists$ a nbhd $B_\delta(x_0)$ for which $f(x) > \mu$.

22 a. Suppose $y_0 = \lim_{x \to x_0} g(x)$ exists and is an interior point of $D_f$ and $f$ is continuous at $y_0$.

Show $\lim_{x \to x_0} (f \circ g)(x) = f(y_0)$.

Rmk This result is the same as $\lim_{x \to x_0} f(g(x)) = f(\lim_{x \to x_0} g(x)) = f(y_0)$.

That is we can pass the limit to the argument of a continuous function.

Pf. $y_0$ is an interior point of $D_f \Rightarrow \exists\, \eta > 0$ s.t. $B_\eta(y_0) \subset D_f$.

Those if $y \in B_\eta(x_0)$ then $f(y)$ exists. Fix $\varepsilon > 0$

~~By continuity of f at $y_0$, $|y - y_0| < \eta \Rightarrow |f(y) - f(y_0)| < \varepsilon$~~

By continuity of $f$ at $y_0$, $\exists\, \eta > 0$ s.t. $|y - y_0| < \eta \Rightarrow |f(y) - f(y_0)| < \varepsilon$

Now let $y = g(x)$.

Since $\lim_{x \to x_0} g(x) = y_0 \Rightarrow \exists\, \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |g(x) - y_0| < \eta$

$\Rightarrow |y - y_0| < \eta$. since $y = g(x)$

$\Rightarrow |f(y) - f(y_0)| < \varepsilon$

$\Rightarrow |f(g(x)) - f(y_0)| < \varepsilon$.

$\Rightarrow \lim_{x \to x_0} f(g(x)) = f(y_0)$.