Math 481 - Hwk #4

Section 2.3 # 1, 6, 22

Section 2.5 # 1, 2, 18a

① Prove f is differentiable at x_0 iff $\lim_{x\to x_0} \frac{f(x)-f(x_0)-m(x-x_0)}{x-x_0} = 0$.

for some constant m, in which case $f'(x_0)=m$

$\Rightarrow$ Assume f is differentiable. Thus $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists

Let $m = f'(x_0)$. Then note $\lim_{x\to x_0} m = m$.

$$\Rightarrow \lim_{x\to x_0} \frac{f(x)-f(x_0)-m(x-x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} - \lim_{x\to x_0} m\frac{(x-x_0)}{x-x_0}$$

$$= 0$$

This line is true since both limits on RHS exist.

$\Leftarrow$ Assume $\lim_{x\to x_0} \frac{f(x)-f(x_0)-m(x-x_0)}{x-x_0} = 0$

Then

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} - m + m \right]$$

$$= \lim_{x\to x_0} \left[\frac{f(x)-f(x_0)-m(x-x_0)}{x-x_0} + m \right]$$

$$= \lim_{x\to x_0} \frac{[f(x)-f(x_0)-m(x-x_0)]}{x-x_0} + \lim_{x\to x_0} m$$

since both limits exist

$$= m.$$

$\Rightarrow$ f is differentiable at x_0.

(6.) Suppose $f'(0)$ exists and $f(x+y) = f(x) f(y)$ for all x and y. Prove f' exists for all x

Let's first understand what $f(x+y) = f(x)f(y)$ implies.

Suppose both x and $y = 0 \Rightarrow f(0+0) = f(0) \cdot f(0)$

$\Rightarrow f(0) = \left(f(0)\right)^2 \Rightarrow f(0) = 0$ or $f(0) = 1$

If $f(0) = 0 \Rightarrow f(0+y) = f(0) \cdot f(y) = 0$

$\Rightarrow f(y) = 0 \ \forall y$

$\Rightarrow f' = 0 \ \forall y$ exists

If $f(0) = 1$, then consider

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) f(h) - f(x)}{h}$$

$$= \lim_{h\to 0} f(x) \cdot \frac{(f(h) - 1)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

since $f'(0)$ exists

$$= f(x) f'(0)$$

$\Rightarrow f'(x)$ exists $\forall x$.

(22) Give an example of a function f s.t. f' exists on (a,b) and has a jump discontinuity at $x_0 \in (a,b)$ or show no such function exists.

Pf/ no such function exists. Indeed assume that $f(x)$ is diff and ~~has~~ f' has a jump discontinuity. In that case the one-sided derivatives $f'_+(x_0)$ and $f'_-(x_0)$ are not equal, which contradicts that f' exists on all of (a,b).

Section 2.5

(1.) Let $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show f has derivatives of all orders and every Taylor polynomial of f about 0 is identically 0.

First note that f is continuous at 0 by using that $\lim_{x \to 0} f(x) = 0$.
It is differentiable at 0, by computing $\lim_{x \to 0} \frac{e^{-1/x^2} - f(0)}{x} = 0$ using L'Hospital's Rule. In fact if we simply calculate $f'(x)$ for $x \neq 0$ and then take limit as $x \to 0$, we find that $f'(x)$ is composed of the product of e^{-1/x^2} with a rational function (ratio of two polynomials)
Since $e^{-1/x^2} \to 0$ faster than any ~~polynomial~~ rational function $\to \infty$

$\Rightarrow f'(0) = 0$.
Similar arguments hold for any order derivative.

$\Rightarrow f^{(n)}(0) = 0$

$\Rightarrow T_n(x) = 0 \quad \forall n.$

This problem shows that a perfectly well behaved function like $f(x)$ may not be well approximated by a polynomial.

(2) Suppose $f^{(n+1)}(x_0)$ exists and let T_n be the n^{th} Taylor polynomial of f about x_0.

Let $E_n(x) = \begin{cases} \dfrac{f(x) - T_n(x)}{(x-x_0)^n} & x \in D_f - \{x_0\} \\ 0 & x = x_0 \end{cases}$

Show that $E_n(x)$ is differentiable at x_0 and find $E'(x_0)$

First note that since $f^{n+1}(x_0)$ exists, by Taylor's Theorem

$$f(x) = T_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \quad \text{for } c \text{ between } x_0 \text{ and } x$$

provided that $f^{(n+1)}$ exists in a nbhd of x_0. But this is not actually guaranteed by the assumption of differentiability at x_0. Indeed, differentiability at a point does not guarantee differentiability in a nbhd of that pt. So we need an alternate method.

Observe that since $f^{n+1}(x_0)$ exists, by Theorem 2.5.1,

$$\lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x-x_0)^{n+1}} = 0. \quad (*)$$

Now let us consider finding the derivative of $E_n(x)$ at the point x_0.

$$E_n'(x_0) = \lim_{x \to x_0} \frac{E_n(x) - E(x_0)}{x - x_0}$$

$\swarrow$ $E(x_0) = 0$

$$= \lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x-x_0)^{n+1}}$$

$$= \lim_{x \to x_0} \frac{f(x) - T_n(x) - \frac{f^{(n+1)}(x_0)(x-x_0)^{n+1}}{(n+1)!} + \frac{f^{(n+1)}(x_0)(x-x_0)^{n+1}}{(n+1)!}}{(x-x_0)^{n+1}}$$

$$= \lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x-x_0)^{n+1}} + \frac{f^{(n+1)}(x_0)(x-x_0)^{n+1}}{(n+1)!\,(x-x_0)^{n+1}}$$

$\swarrow$ by $(*)$ since both terms on RHS have limits

$$= \frac{f^{(n+1)}(x_0)}{(n+1)!}$$

(18a). If $\Delta^0 f(x) = f(x)$, $\Delta f(x) = f(x+h) - f(x)$

$\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$ $n \geq 1$ Show by induction

that $\Delta^n [c_1 f_1(x) + \cdots + c_k f_k(x)] = c_1 \Delta^n f_1(x) + \cdots + c_k \Delta^n f_k(x)$.

Pf, for simplicity of notation let's take $k=2$, and then recognize for any finite k, the proof is easily extendable by the associative law. We will use m as an intermediary inductive variable.

Base Case $n=1$ (6)

LHS: $\Delta[c_1 f_1(x) + c_2 f_2(x)] = c_1 f_1(x+h) - c_1 f_1(x) + c_2 f_2(x+h) - c_2 f_2(x)$

$= c_1 [f_1(x+h) - f_1(x)] + c_2 [f_2(x+h) - f_2(x)]$

$= c_1 \Delta f_1 + c_2 \Delta f_2$

$=$ RHS ✓

$\Rightarrow$ base case is true.

Assume true for $n=m$, that is $\Delta^m [c_1 f_1(x) + c_2 f_2(x)] = c_1 \Delta^m f_1(x) + c_2 \Delta^m f_2(x)$

Then for $n=m+1$

$$\Delta^{m+1}[c_1 f_1(x) + c_2 f_2(x)] = \Delta[\Delta^m (c_1 f_1(x) + c_2 f_2(x))]$$

$$= \Delta[c_1 \Delta^m f_1(x) + c_2 \Delta^m f_2(x)]$$ } by inductive step

$$= c_1 \Delta^{m+1} f_1(x) + c_2 \Delta^{m+1} f_2(x)$$

Thus the formula holds for $n=m+1$.

$\Rightarrow$ by induction the formula holds $\forall n \geq 1$.